Laplace Transformation and its applications

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Abstract: In this paper, we are interested to discuss about properties and applications of Laplace transform in various fields. Also we discuss Laplace transform has the master techniques used by researchers, scientists and mathematicians to find results of their problems. In this paper we will study to solve research problems by using Laplace transform. The motive of this paper is that a scientific review on properties and applications of Laplace transform. This paper also includes the formulation of Laplace transform of important functions like the periodic functions, Unit Impulse function.

Keywords: Laplace transforms, Properties, Differential equation

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I. Introduction

In this paper overview of properties of Laplace transform with definition and its application in engineering and applied science. The Laplace transform is integral transform which is denoted by L[f(t)]. The solution of linear, ordinary differential equation with constant coefficients such as the third order equation $af'''(t) + bf''(t) + cf'(t) + df(t) = g(t)$ can be solved by first obtaining the general form for the expression $f(t)$. This general form will contain a integration constants whose values can be found by appropriate boundary conditions. A various systematic way of solving these equations is to use the Laplace transform which transform the differential equation into an algebraic equation and has the added incorporate advantage the boundary conditions from the beginning. Furthermore, if (t) represents function with discontinuities, other methods fail where Laplace transform method can succeed.

Laplace transform techniques also provide powerful in various fields of technology such as control theory, population growth and decay problems where knowledge of the system transfer function is important and at which Laplace transform comes into its own.

2. Definition of Laplace transform

The Laplace transform of the function (t) for all $t \geq 0$ is defined as [1-5]

$$L[f(t)] = \int_0^\infty f(t)e^{-st}dt = F(s) \quad ..... \qquad (1)$$

Where L is Laplace transform operator. The Laplace transform of the function (t) for all $t \geq 0$ exist if $f(t)$ is exponential order and piecewise continuous. These are only sufficient conditions for the existence of Laplace transform of the function (t).

3. Properties of Laplace transform [2]

3.1 Linearity Property

$If\ [f(t)] = \bar{f}(s)\ \&\ L[g(t)] = \bar{g}(s)$ then

$[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$ where $a\ and\ b$ are arbitrary constants.

3.2 First Shifting Property

$If\ L[f(t)] = \bar{f}(s), then\ L[e^{at}f(t)] = \bar{f}(s-a)$

3.3 Convolution Theorem

$If\ L^{-1}[\bar{f}(s)] = f(t)\ \&\ L^{-1}[\bar{g}(s)] = g(t)\ then$

$$L^{-1}[\bar{f}(s) * \bar{g}(s)] = \int_0^t f(u)g(t-u)du$$

3.4 Laplace transform of Derivative

$[f'(t)] = sL[f(t)] - f(0),$

$[f''(t)] = s^2L[f(t)] - sf(0) - f'(0),$

$[f'''(t)] = s^3L[f(t)] - s^2f(0) - sf'(0) - f''(0),$ and so on.

3.5 Laplace transform of Integrals

$If\ L[f(t)] = \bar{f}(s), then$

$$L\left\{\int_0^t f(u)du\right\} = \frac{1}{s}\bar{f}(s)\ \ \&\ \int_0^t f(u)du = L^{-1}\left(\frac{\bar{f}(s)}{s}\right)$$

3.6 Multiplication by t^n

$$If\ L[f(t)] = \bar{f}(s), then$$

$$(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}(\bar{f}(s)), n \in Z^+$$

3.7 Division by t

$$If\ L[f(t)] = \bar{f}(s), then$$

$$L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)ds$$

3.8 Laplace transform of Unit step function is

$(u(t-a)) = \frac{1}{s}e^{-as}$ where $u(t-a) = \begin{cases} 0 & t < a \\ 1 & \geq a, a \geq 0 \end{cases}$

3.9 Second shifting Theorem

$$If\ L[f(t)] = \bar{f}(s), then$$

$$L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$$

3.10 Laplace transform of unit Impulse function

$[\delta(t-a)] = e^{-as}$ where $\delta(t-a) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases}$

3.11 Laplace transform of Periodic function

$$L[f(t)] = \frac{1}{1 - e^{sT}}\int_0^T e^{-st}f(t)dt$$

4. Application

4.1. Population Growth problem

The growth of population (growth of a species, an organ, or a plant, or a cell) is can be written as first order linear ordinary differential equation[6-15].

$\frac{dP}{dt} = \alpha P$ .... (2)

with the initial condition as

$(t_0) = P_0$......... (3)

Where α is a positive real number, P is the amount of population at time t and P_0 is the initial population at time t_0. Equation (2) is also known as Malthusian law of population growth.

The decay problem of the substance is defined mathematically by the first order linear ordinary differential equation [12,14-15]

$\frac{dP}{dt} = -\alpha P$ .... (4)

with the initial condition as

$(t_0) = P_0$......... (5)

Where P is the amount of substance at time t, α is a positive real number and P_0 is the initial population at time t_0.

In equation (4), the negative sign in the right side is taken as mass of the substance is decreasing with time and so derivative $\frac{dP}{dt}$ must be negative.

In this, we present Laplace transform for population growth problem given by (2) and (3)

By applying Laplace transform on both sides of (2),

$L[\frac{dP}{dt}] = \alpha L[P(t)]$...... (6)

Now applying the property, Laplace transform of derivative of function, on (6), we have

$s[P(t)] - P(0) = \alpha L[P(t)]$........ (7)

Using (3) in (7) and on simplification, we have

$(s-\alpha)[P(t)] = P_0$

$=> [P(t)] = \frac{P_0}{(s-\alpha)}$............... (8)

By operating inverse Laplace transform on both sides of (8), we have

$(t) = L^{-1}[\frac{P_0}{(s-\alpha)}]$

$=> (t) = P_0 e^{\alpha t}$............ (9)

This is required amount of population at time t.

Similarly by applying Laplace transform for decay problem given in (4) and (5)

$=> (t) = P_0 e^{-\alpha t}$............ (10)

This is required amount of substance at time t.

Application-1:

The population of town grows at a rate proportional to the number of people presently living in the town. If after five years, the population has doubled, and after ten years the population is 200000, estimate the number of people initially living in the town.

This problem mathematically written as

$$\frac{dP(t)}{dt} = \alpha P(t)\ldots\ldots.. \quad (11)$$

Where P denote the number of people living in the town at any time t and α is the constant of proportionality.

Consider P_0 is the number of people initially living in the town at t=0.

Then by Laplace transform applying in (11)

$$L[\frac{dP}{dt}] = \alpha L[P(t)]\ldots\ldots \quad (12)$$

Now applying the property, Laplace transform of derivative of function, on (12)

$$L[P(t)] = \frac{P_0}{(s-\alpha)}$$

$$=> (t) = P_0 e^{\alpha t}\ldots\ldots\ldots.. \quad (13)$$

Now at t=5,$(t) = 2P_0$, use in (13) we get

$=>2P_0 = P_0 e^{\alpha t}$

$=> 2=e^{\alpha 5}$

$$=> \alpha = 0.13862943611 \ldots\ldots\ldots\ldots\ldots\ldots \quad (14)$$

Now using the condition t=10, at that time P=200000, use in (13) we have

$=>200000=P_0{}^{(0.13862943611)*10}$

$$=> P_0 = 50000\ldots\ldots\ldots\ldots.. \quad (15)$$

This is initially population in the town.

Application: 2

A phosphorus substance is known to decay at a rate proportional to the amount present. If initially there is 1000 milligrams of the phosphorus substance present after five hours it is observed that the phosphorus substance has lost 20 percent of its original mass, find the life of the phosphorus substance.

The problem can be written in mathematical form as

$$\frac{dP(t)}{dt} = -\alpha P(t)\ldots\ldots\ldots\ldots\ldots\ldots \quad (16)$$

Where P denote the amount of phosphorus substance at time t and α is the constant of proportionality. Consider P_0 is the initial amount of the phosphorus substance at time $t = 0$.

By applying Laplace transform in (16)

$$L\,[\frac{dP}{dt}] = -\alpha L[P(t)]\ldots\ldots \quad (17)$$

Now applying the property, Laplace transform of derivative of function, and at $t = 0$,

$P = P_0 = 1000$, Use in (17), we have

⇨ $L[P(t)] = \frac{1000}{(s+\alpha)}$

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$s[P(t)] - 1000 = -\alpha L[P(t)]$

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⇨ $P(t) = L^{-1}[\frac{1000}{(s+\alpha)}]$

⇨ $(t) = 1000e^{-\alpha t}$………….. (18)

Now at, t=5, the phosphorus substance has lost 20 percent of its original mass 1000 mg so $(5) = 1000 - 200 = 800$, Using this in (18), we have

$800 = 1000e^{-5\alpha}$

=> 0.8= $e^{-5\alpha}$

$$=> \alpha = -\frac{1}{5}\log(0.8)$$

=> $\alpha = 0.04462871026$…………….. (19)

Here we want to find time $P = \frac{P_0}{2} = 500$ so from (18), we have

$500 = 1000e^{-(0.04462871026)t}$

$$=> t = \frac{\log(0.5)}{-0.04462871026}$$

=> $t = 15.5314186$

15.53 hours required half time of the phosphorus substance.

4.2 Problem in Mechanical Engineering

Vibrating Mechanical systems: If we discuss the suspension system of the car the mass is an important, damper and springs used to join the body of the car. Mechanical systems may be used to model many situations, and involve three basic elements: masses (mass M measured in kg), dampers (damping coefficient B measured in Nsm^{-1}. The associated variables are force F(t) (measured in N) and displacement Y(t) (measured in M).

Consider Mass-damper- spring system using Newton's Hooke's Law therefore differential equation given by If M=1, B=4, K=10 & $(t) = 5sinwt$

$$Y''(t) + 4Y'(t) + 10Y(t) = 5sinwt$$

If we apply Laplace transform then we have

$$[Y''(t)] + 4L[Y'(t)] + 10L[Y(t)] = 5L[sinwt]$$

⇨ $(s^2 + 4s + 10)L[Y(t)] = [sY(0) + Y'(0) + 4Y(0)] + \frac{5w}{s^2+w^2}$

⇨ If consider initial condition $Y(0) = 0 = Y'(0)$

⇨ $L[Y(t)] = \frac{5w}{(s^2+w^2)((s^2+4s+10)}$

⇨ $Y(t) = L^{-1}[\frac{5w}{(s^2+w^2)((^2+4s+10)}]$

⇨ $Y(t) = L^{-1}[\frac{\frac{-20s}{97}+\frac{45}{97}}{s^2+1} + \frac{\frac{20s}{97}+\frac{35}{97}}{s^2+4s+10}]$

⇨ $Y(t) = \frac{-20}{97}cost + \frac{45}{97}sint + \frac{20}{97}e^{-2t}cos\sqrt{6}t - \frac{5}{97\sqrt{6}}e^{-2t}sin\sqrt{6}t$

4.3 Application in Electrical Engineering

Consider simple electric circuit where R-resistance, L-inductance, C-capacity and E-electromotive power of voltage in a series. A switch is connected in the circuit.

Then by Kirchhoff law

$L\frac{dI}{dt} + RI + \frac{Q}{c} = E$……….

Consider resistor of 25 ohms inductance of 5 henry, capacitor of 0.05 farad at t=0, the change on the current and capacitor in the circuit is zero. What is current and charge at any time t>0.

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Consider I and Q are instantaneous current and charge respectively at time t,

$$5\frac{dI}{dt} + 25I + 20Q = 100$$

$$\frac{d^2Q}{dt^2} + 5\frac{dQ}{dt} + 4Q = 20$$

By applying Laplace transform we have

$$[\frac{d^2Q}{dt^2}] + L[5\frac{dQ}{dt}] + 4L[Q] = 20L[1]$$

$$(s^2 + 5s + 4)(Q) = \frac{20}{s}$$

$$(Q) = \frac{20}{s(s^2 + 5s + 4)}$$

$$Q = L^{-1}[\frac{20}{(s^2 + 5s + 4)}]$$

$$Q = L^{-1}[\frac{5}{s} + \frac{-5s - 25}{(s^2 + 5s + 4)}]$$

For t > 0 required the current and charge expression is

$$Q = 5 - 5^{(-5/2)t} \cosh\frac{3}{2}t - \frac{25}{3}^{(-5/2)t} \sinh\frac{3}{2}t$$

$$I = \frac{dQ}{dt} = \frac{40}{3}^{(-5/2)t} \sinh\frac{3}{2}$$

4.4 Application in Nuclear Physics

If we consider first order linear differential equation

$$\frac{dP}{dt} = -\alpha P$$

This equation is the fundamental relationship describing radioactive decay, where P = P(t) represents the number of remaining radioactive substances at time t and α is the decay constant.

We can solve this as earlier decay problem mentioned.

5. Conclusion:

Through this paper we shows the applications of Laplace transform in various fields like Engineering (Mechanical, Electrical etc.), Physics, Growth and decay problems . Laplace transform is a very effective tool to solve very complex problem of most of engineering and science field. In these days there is tremendous use of Laplace transform to find the solution of different problems.

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