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Hop domination number of Lollipop graph, Barbell graph and Book graph

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ABSTRACT

Let G be a Lollipop graph or Barbell graph or Book graph. A set $S_h \subseteq$V(G) is a hop dominating set of G if for all v in $V - S_h$, there exists u in S_h such that $d(u,v) = 2$. The minimum cardinality of a hop dominating set of G is called the hop domination number of G and is denoted by $\gamma_h(G)$. In this paper, we have discussed about the hop domination number of Lollipop graph, Barbell graph and Book graph.

KEYWORDS: hop-domination, hop-domination number, Lollipop graph, Barbell graph and Book graph.

I. Introduction

[3]A set $S_h \subseteq$V(G) is a hop dominating set of G if for all v in $V - S_h$, there exists u in S_h such that $d(u,v) = 2$. The minimum cardinality of a hop dominating set of G is called the hop domination number of G and is denoted by $\gamma_h(G)$. [7] The Lollipop graph is the graph obtained by joining a complete graph to a path graph with a bridge. It is denoted by $L_{m,n}$ or $L(m,n)$. The hop-domination number of $L_{m,n}$ is denoted by $\gamma_h(L_{m,n})$. The generalised Lollipop Graph $\left(\boldsymbol{L_{m,n}}\right)$ is

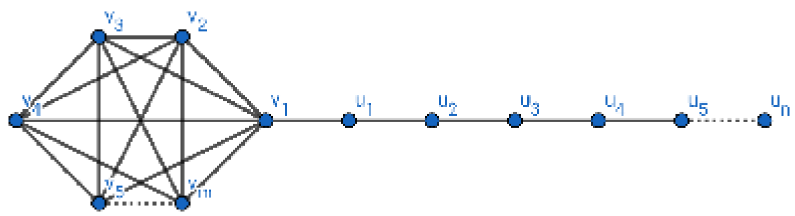


Fig. 1.1

Let us denote the vertices of a lollipop graph as two distinct sets:

(i) Refer the vertices of the complete graph k_m as $\{v_1, v_2, \dots v_m\}$ and

(ii) The Vertices of the Path graph p_n as $\{p_1, p_2, \dots p_n\}$

∴ The vertices of $L_{m,n}$ are

$$V(L_{m,n}) = V(k_m) \cup V(p_n)$$

$$= \{v_1, v_2, \dots v_m, p_1, p_2, \dots p_n\}.$$

[6]The book graph is the Cartesian product of $S_{m+1} \times P_2$ where S_{m+1} is a star graph and P_2 is a path graph on two vertices. It is denoted by $B_{m,n}$. The hop-domination number of $B_{m,n}$ is denoted by $\gamma_h(B_{m,n})$.

[8]A (2,n) Barbell graph is obtained by connecting two copies of a complete graph $n(K_n)$ by a bridge. It is denoted by $B(n,n)$ or $B(K_n, K_n)$. The hop domination number of $B(K_n, K_n)$ is denoted by $\gamma_h(B(K_n, K_n))$. Generalised Barbell Graph $\left(B_{n,n}\right)$ is

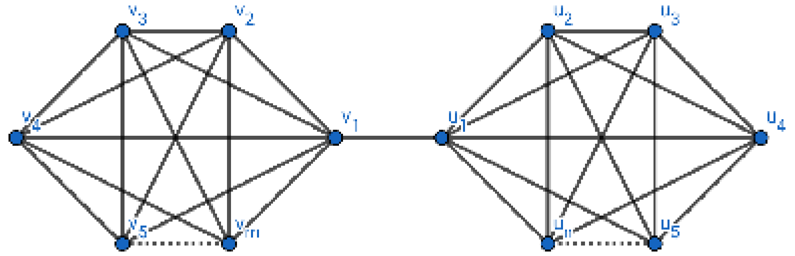


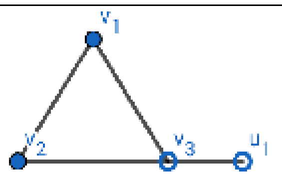
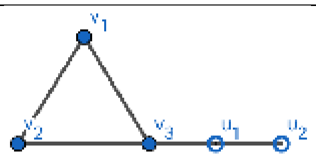
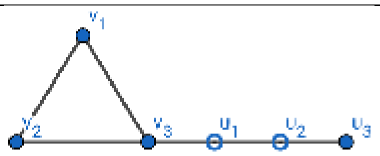
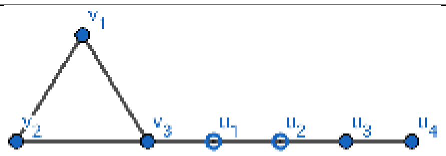
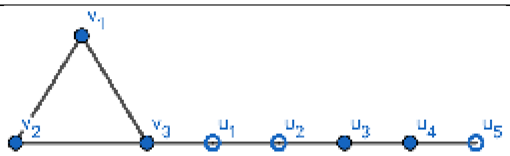
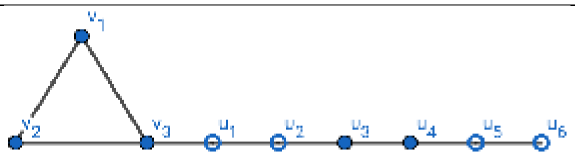
Fig. 1.2

Theorem 1.1(Ref [11] p.546): A dominating set D of a graph G is minimal iff for each vertex $v \in D$, one of the following conditions satisfied,

(i) There exists a vertex $u \in V - D \; such \; that \; N(u) \cap D = \{v\}$

(ii) v is an isolated vertex in D.

2. Diagrammatic discussion on Hop domination Number of Lollipop Graph, Book Graph and Barbell Graph

S.No.	**Lollipop Graph $(L_{m,n}), m = 3$**	**Graph**	$\gamma_h(G)$
1	n=1, $L_{3,1}$		2
2	n=2, $L_{3,2}$		2
3	n=3, $L_{3,3}$		2
4	n=4, $L_{3,4}$		2
5	n=5, $L_{3,5}$		3
6	n=6, $L_{3,6}$		4

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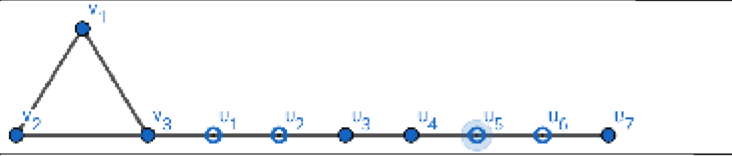
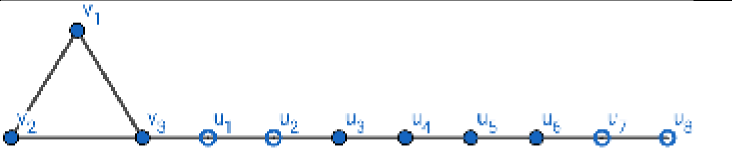
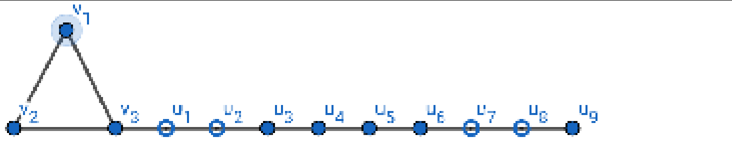
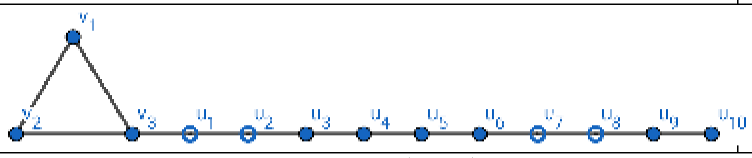
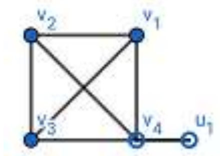
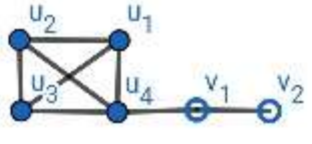
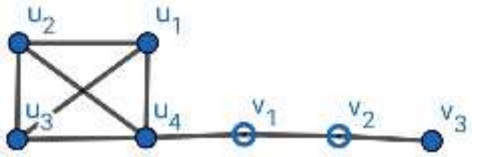
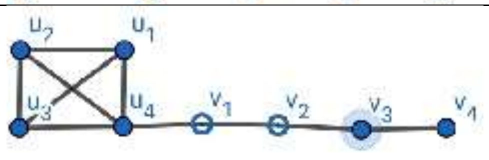
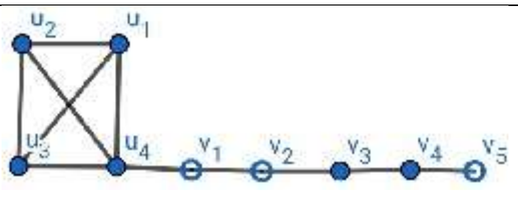
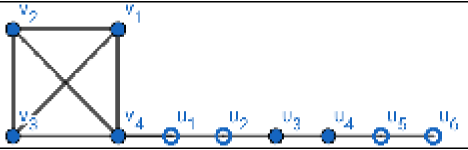
7	n=7, $L_{3,7}$		4
8	n=8, $L_{3,8}$		4
9	n=9, $L_{3,9}$		4
10	n=10, $L_{3,10}$		4

Table 2.1: Lollipop Graph $(L_{m,n}), m = 3$

S.No.	Lollipop Graph $(L_{m,n}), m = 4$	Graph	$\gamma_h(G)$
1	n=1, $L_{4,1}$		2
2	n=2, $L_{4,2}$		2
3	n=3, $L_{4,3}$		2
4	n=4, $L_{4,4}$		2
5	n=5, $L_{4,5}$		3
6	n=6, $L_{4,6}$		4

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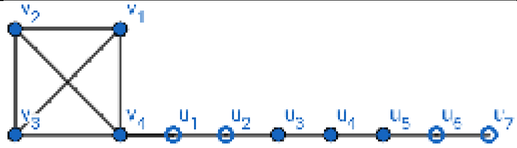
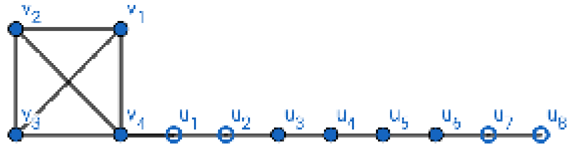
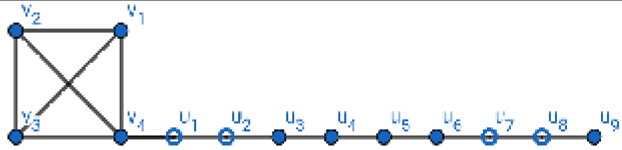
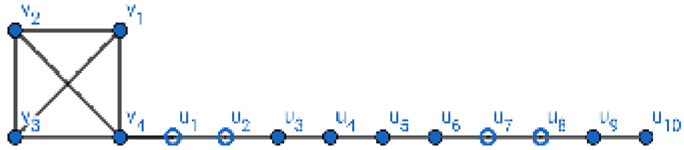
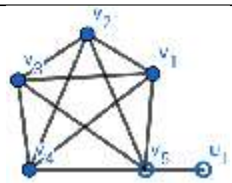
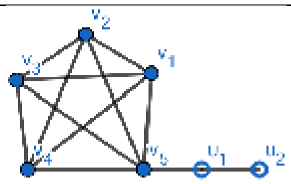
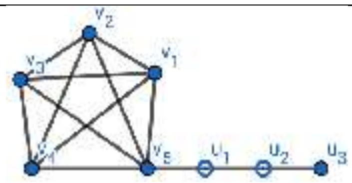
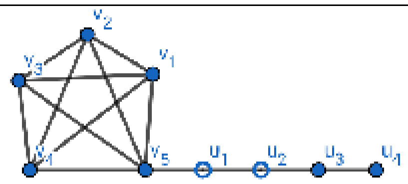
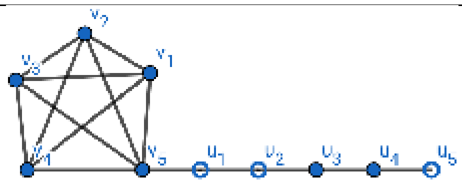
7	n=7, $L_{4,7}$		4
8	n=8, $L_{4,8}$		4
9	n=9, $L_{4,9}$		4
10	n=10, $L_{4,10}$		4

Table 2.2: Lollipop Graph $(L_{m,n}), m = 4$

S.No.	Lollipop Graph $(L_{m,n}), m = 5$	Graph	$\gamma_h(G)$
1	n=1, $L_{5,1}$		2
2	n=2, $L_{5,2}$		2
3	n=3, $L_{5,3}$		2
4	n=4, $L_{5,4}$		2
5	n=5, $L_{5,5}$		3

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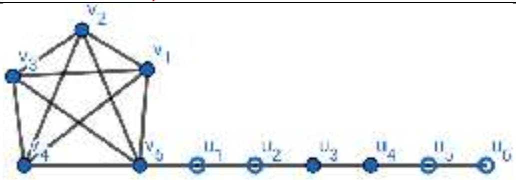
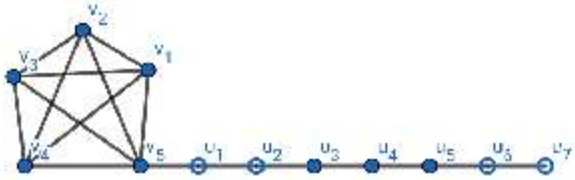
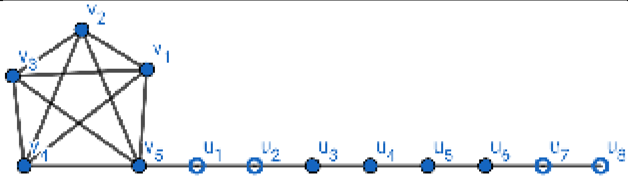
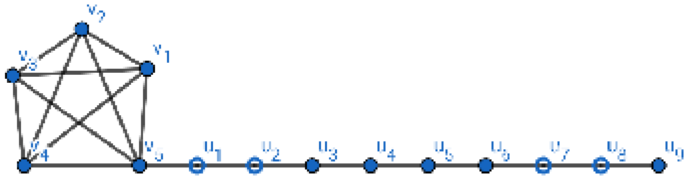
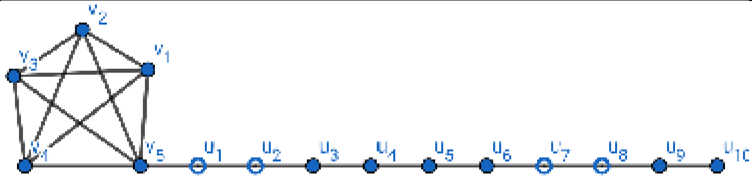
6	n=6, $L_{5,6}$		4
7	n=7, $L_{5,7}$		4
8	n=8, $L_{5,8}$		4
9	n=9, $L_{5,9}$		4
10	n=10, $L_{5,10}$		4

Table 2.3: Lollipop Graph $(L_{m,n}), m = 5$

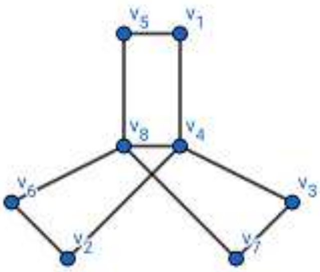
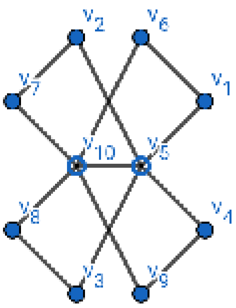
S.No.	Hop dominating set	Graph	$\gamma_h(G)$
1	$S_h = \{v_4, v_8\}$		2
2	$S_h = \{v_5, v_{10}\}$		2

Table 2.4: Book Graph $(B_{m,n})$

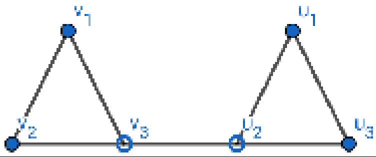
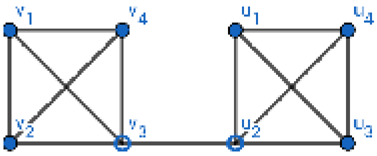
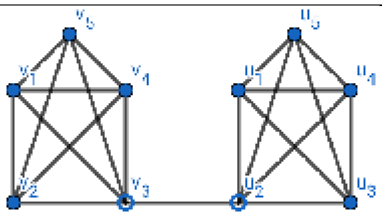
S.No.	Barbell Graph $(B_{n,n})$	Graph	$\gamma_h(G)$
1	$B(3,3)$ or $B(K_3, K_3)$		2
2	$B(4,4)$ or $B(K_4, K_4)$		2
3	$B(5,5)$ or $B(K_5, K_5)$		2

Table 2.5: Barbell Graph $(B_{n,n})$

3. Results on Hop domination number of Lollipop Graph $(L_{m,n})$, Barbell Graph$(B_{n,n})$ and Book Graph $(B_{m,n})$

Theorem 3.1: The hop domination of a lollipop graph $L_{m,n}$ is given by

$$\gamma_h(L_{m,n}) = \begin{cases} 2k+2 & if\ n = 6k + r, 0 \le r \le 4 \\ 2k+3 & if\ n = 6k+5 \end{cases}$$

Proof:

Let S_h be the hop dominating set of $L_{m,n}$. The minimality of the set S_h follows from theorem (1.1) by using the contrary of the theorem.

If S_h is not a minimal hop dominating set then $\exists\ v\ \in S_h : S_h{}' = S_h - \{v\}$ is a hop dominating set of $L_{m,n}$.

Therefore, $\forall\ u \in N'\ [V], \exists\ v'\ \in S_h - \{v\}, v' \in N'[V]$

Case (i): $n = 6k$

Let $S_h = \{p_{6m-5}, p_{6m-4}, \dots p_{n-1}, p_n\ /\ m = 1,2, \dots \dots . k\}$

Subcase (a): If $V = \{p_{6m-5}\ /m = 1,2, \dots \dots . k\}$ then either $N[V_m]$ or atleast one vertex of the form $p_{6m-7} or\ p_{6m-3}$ are not hop dominated by any vertex in $S_h{}'$.

Subcase (b): If $V = \{p_{6m-4}\ /m = 1,2, \dots \dots . k\}$ then either $[V_m]$ or atleast one vertex of the form $p_{6m-6} or\ p_{6m-2}$ are not hop dominated by any vertex in $S_h{}'$.

Subcase (c): If $V = \{p_n\}$ or $\{p_{n-1}\}$ then no vertex in $S_h{}'$ hop dominates p_n and p_{n-1}. Therefore, $N'[S_h{}'] \neq V$. So $S_h{}'$ is not a hop dominating set. Hence S_h is minimum.

Case (ii): If $n = 6k + 1$

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Let $S_h = \{p_{6m-5}, p_{6m-4}, p_{n-2}, p_{n-1} \ / \ m = 1,2, \dots \dots k\}$. If $v = p_{6m-5}$ or p_{6m-4} or p_{n-1}, the minimality of S_h follows from Case (i) or else if $v = p_{n-2}$ there exist no vertex in S_h' hop dominating p_n. Therefore, $S_h{}'$ is not a hop dominating set. Hence S_h is minimum.

Case (iii): If $n = 6k + 2$

Let $S_h = \left\{\{p_{6m-5}, p_{6m-4}, p_{n-2}, p_{n-1}\right\}$. If $v = p_{6m-5}$ or p_{6m-4}, the minimality of S_h follows from Case (i) or else if $v = p_{n-2}$, it follows from Case (ii). If $v = p_{n-3}$, there exist no vertex in S_h' hop dominating p_{n-1}. Hence $S_h{}'$ is not a hop dominating set. Thus S_h is minimum.

Case (iv): If $n = 6k + 3 \ and \ n = 6k + 4$.

Let $S_h = \{p_{6m-5}, p_{6m-4}, p_{n-2}, p_{n-3} \ /m = 1,2, \dots \dots k\}$. The minimality of S_h follows from Case (iii). Therefore, For each m, $1 \le m \le k$ ∃ p_{6m-5} & p_{6m-4} in S_h & ∃ two p_i's in S_h independent of m in $V - S_h$. Therefore, $|S_h| = 2k + 2$. Thus $\gamma_h(L_{m,n}) = 2k + 2$ if $n = 6k + r, 0 \le r \le 4$.

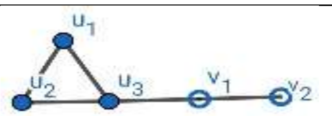
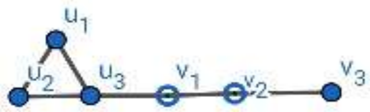
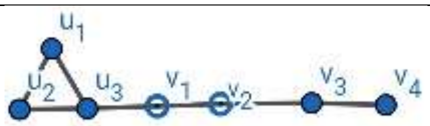
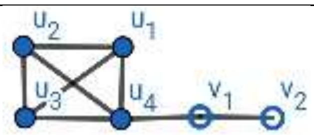
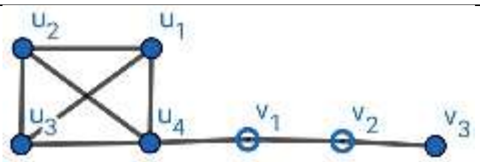
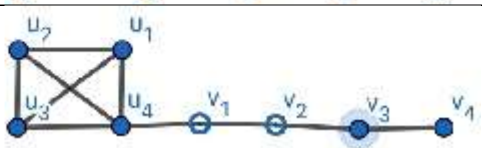
Case (v): If $n = 6k + 5$

Let $S_h = \{p_{6m-5}, p_{6m-4}, p_n \ /m = 1,2, \dots \dots k + 1\}$. The minimality of S_h follows from Case (i). Here for each m, $1 \le m \le k + 1$ there exists p_{6m-5} and p_{6m-4} in S_h and there exists p_n in S_h independent m. therefore, $|S_h| = 2(k + 1) + 1 = 2k + 3$.

Thus $\gamma_h(L_{m,n}) = 2k + 3$ if $n = 6k + 5$.

Theorem 3.2: $\gamma_h(L(m, 2)) = \gamma_h(L(m, 3)) = \gamma_h(L(m, 4)) = 2$.

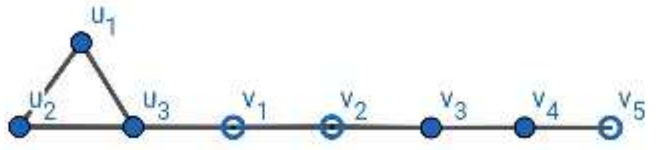
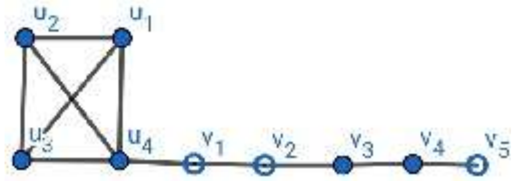
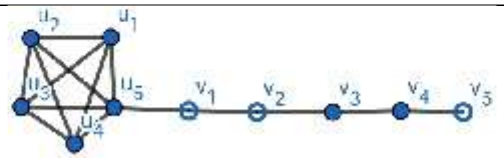
Proof:

L(3,2)	$S_h = \{v_1, v_2\}$	$\gamma_h(L(3,2))$=2	
L(3,3)	$S_h = \{v_1, v_2\}$	$\gamma_h(L(3,3))$=2	
L(3,4)	$S_h = \{v_1, v_2\}$	$\gamma_h(L(3,4))$=2	
L(4,2)	$S_h = \{v_1, v_2\}$	$\gamma_h(L(4,2))$=2	
L(4,3)	$S_h = \{v_1, v_2\}$	$\gamma_h(L(4,3))$=2	
L(4,4)	$S_h = \{v_1, v_2\}$	$\gamma_h(L(4,4))$=2	

In general, let $S_h = \{v_1, v_2\}$. Now we have to prove the minimality of S_h. Suppose if S_h is not minimum, the proper subset $S'_h\ of\ S_h$ must be the hop dominating set of $L(m,n)$, where m=3,4,5,… and n=2,3,4. For n=2, If $v \in S_h\ that\ is\ either\ v = v_1\ or\ v = v_2$. If $v = v_1$, then $S'_h = S_h - v = v_2$, the vertices $u_1, u_2, \ldots, u_{m-1}$ are not hop dominated. Hence S'_h is not a minimal hop dominating set. Else if $v = v_2$, then $S'_h = S_h - v = v_1$, the vertices $u_m, v_2\ and\ v_4$ are not hop dominated. Hence S'_h is not a minimal hop dominating set. Hence a proper subset of S_h is not minimal hop dominating set. Thus S_h is the minimum. Hence $\gamma_h\big(L(m,2)\big) = 2$. The proof follows as the above for $\gamma_h\big(L(m,3)\big) = \gamma_h\big(L(m,4)\big) = 2$.

Theorem 3.3: $\gamma_h\big(L(m,5)\big) = 3, m = 3,4,5, \ldots$

Proof:

L(3,5)	$S_h = \{v_1, v_2, v_5\}$	$\gamma_h\big(L(3,5)\big)$=3	
L(4,5)	$S_h = \{v_1, v_2, v_5\}$	$\gamma_h\big(L(4,5)\big)$=3	
L(5,5)	$S_h = \{v_1, v_2, v_5\}$	$\gamma_h\big(L(5,5)\big)$=3	

In general, Let $S_h = \{v_1, v_2, v_5\}$ is a hop dominating set of $L(m,5)$. Now we have to prove the minimality of S_h. Suppose if S_h is not minimum, the proper subset $S'_h\ of\ S_h$ must be the hop dominating set of $L(m,5)$, where m=3,4,5,…. If $v \in S_h\ that\ is\ either\ v = v_1\ or\ v = v_2$ or $v = v_5$. If $v = v_1$, then $S'_h = S_h - v = \{v_2, v_5\}$, the vertices $u_1, u_2, \ldots, u_{m-1}$ are not hop dominated. Hence S'_h is not a minimal hop dominating set. If $v = v_2$, then $S'_h = S_h - v = \{v_1, v_5\}$, the vertices u_m, v_2, v_4 are not hop dominated. Hence S'_h is not a minimal hop dominating set. And if $v = v_5$, then $S'_h = S_h - v = \{v_1, v_2\}$, the vertex v_5 is not hop dominated. Hence S'_h is not a minimal hop dominating set. Hence a proper subset of S_h is not minimal hop dominating set. Thus S_h is the minimum. Hence $\gamma_h\big(L(m,5)\big) = 3, m = 3,4,5, \ldots$.

Theorem 3.4: $\gamma_h\big(L(m,n)\big) = m\ iff\ n = 0$.

Proof: If $n = 0$ in $L_{m,n}$, then the graph is a complete graph with m vertices. We know that, $\gamma_h(K_m) = m$. Hence $\gamma_h\big(L(m,n)\big) = m\ if\ n = 0$. Conversely assume $\gamma_h\big(L(m,n)\big) = m$. Now we have to prove that $n = 0$. On the contrary if $n \neq 0$, $\gamma_h\big(L(m,n)\big)$ is given by theorem 3.1. That is, $\gamma_h\big(L(m,n)\big) \neq m$ which contradicts our assumption. Then $n = 0$. Hence $\gamma_h\big(L(m,n)\big) = m\ iff\ n = 0$.

Theorem 3.5: For $m \geq 3$, the hop domination number of book graph B_m is 2. (i.e., $\gamma_h(B_m) = \gamma_h(S_{m+1} \times P_2) = 2$).

Proof:

Let $V(B_m) = \{v_1, \dots, v_m, v_{m+1}, \dots, v_{2m+1}, v_{2m+2}$. Note that v_{m+1} and $v_{2(m+1)}$ refer to the center of these two stars from fig.(ref. table 4) we have given the hop domination set of this graph as follows: $S_h = \{v_{m+1}, v_{2(m+1)}\}$. It is very easy to prove the minimality of this set. Suppose S_h is not minimum this implies that there is a proper subset hop dominating B_m.

If $v \in S_h$ $i.e.$, either $v = v_{m+1}$ or $v = v_{2(m+1)}$. If $v = v_{m+1}, Let\ S_h' = S_h - v$ we have all the vertices that adjacent to v not hop dominating with the vertex $v_{2(m+1)} = S_h - v$ $=S_h'$. Therefore S_h' is not hop dominating set. Similarly for $v = v_{2(m+1)}$.

So S_h chosen in this way ensures that there is no proper subset of S hop dominates B_m.

∴ S_h is minimum. $\gamma_h(B_m) = \gamma_h(S_{m+1} \times P_2) = |S_h| = 2$.

Theorem 3.6: The hop domination number of a Barbell graph $B(N, n)$ given by, $\gamma_h\big(B(N, n)\big) = 2\ if\ \ N = 2$.

Proof: Let $B(2, n)$ be a Barbell graph. We know that, it has $2n$ vertices and $2\binom{n}{2} + 1$ edges. Let $V\big(B(2, n)\big) = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$. We know that, in a Barbell graph, 2 cliques are connected by an edge. Let $v_1, v_2, \dots, v_n$ be the vertices of the first clique and $u_1, u_2, \dots, u_n$ be the vertices of the other clique. These two cliques are connected by an edge from any $v_i\ to\ u_i,\ \ i = 1,2, \dots, n,$ that is the edge between two cliques is

$$\{(v_1, u_1)\ (or)\ \ (v_1, u_2)\ (or) \dots (or)\ (v_1, u_n)$$

$$(v_2, u_1)\ (or)\ \ (v_2, u_2)\ (or) \dots (or)\ (v_2, u_n)$$

$$\vdots$$

$$(v_n, u_1)\ (or)\ \ (v_n, u_2)\ (or) \dots (or)\ (v_n, u_n)\}$$

Without loss of generality, we may choose it as (v_1, u_1). Hence $S_h = \{v_1, u_1\}$ as u_1 is hop dominates all $v_i's$ except v_1 and v_1 is hop dominates all $u_i's$ except u_1. Also v_1 & u_1 hop dominates itself.

Now we have to check the minimality of S_h. Suppose on the contrary, if the above chosen S_h is not minimum, then there exists a proper subset of $S_h'\ of\ S_h,$ hop dominating $B(2, n)$.

If $v \in S_h'$, then either $v = v_1$ $(or)\ v =$ u_1. If $v = v_1$ then $S_h - v = \{u_1\}$, then v_1 not hop dominates u_1 or else if $v = u_1$ then $S_h - v = \{v_1\}$, then u_1 not hop dominates v_1. Hence no proper subset of S_h hop dominates $B(2, n)$. Thus S_h is minimum. Therefore, $\gamma_h\big(B(2, n)\big) = 2$.

Result: $\gamma_h\big(L_{3,n}\big) = \gamma_h(T_{3,n})$.

Proof: Since a cycle with 3 vertices and a clique with 3 vertices are same graphs, $L_{3,n} = T_{3,n}$. Hence the hop domination number of these graphs are also equal.

4. Conclusion

In this paper, we have found the hop domination number of Lollipop graph, Book graph and Barbell graph and derived some theorems on it.

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