TWO DIMENSIONAL ANALYTICAL SOLUTIONS FOR PATHOGENIC POLLUTANTS TRANSPORT IN POROUS MEDIUM

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Abstract: *With the help of Laplace transformation technique, an analytical solution is obtained for two-dimensional advection-dispersion equation in semi-infinite homogeneous porous medium. Considered Exponential form of time-dependent first order decay and seepage velocity. Pathogenic pollutant reaches the groundwater through soil surface and moves vertically down-wards and starts spreading in the horizontal plane. Two separates transformations are introduced.*
Key words: *Advection, dispersion, Integral transforms, Moving coordinates, Duhamel's theorem*

Introduction

The use of natural resources and the large production of wastes in modern civilization often create a hazard to the groundwater quality and already have resultant in many incidents of groundwater contamination in particular by pathogenic . Degradation of groundwater quality can take place over large areas from plane or diffuse sources like deep percolation from intensively farmed fields, or it can be caused by point sources such as septic tank, garbage disposal sites, cemeteries, mine spoils and oil spoils or other accidental entry of contaminants into the underground atmosphere. Another possibility is contamination by line sources of poor quality water, like seepage from contaminated streams or intrusion of salt water, from oceans

Analytical solutions in One and Two-dimensional problems through semi-infinite or finite porous media have been studied by several researchers, like Bastian and Lapidus [3], Banks and Ali [4], Ogata [5], Marino [6] and van Genucheten [7]. These papers are revealing idealistic assumptions, such as porous medium of constant porosity, seepage flow and dispersion. seepage flow and dispersion. In deviating from the above ideal conditions, Shamir and Harleman [2] discussed an analytical solution in two layered porous medium. Banks and Jerasate [8], Rumer [9] and Yadav et al. [10] considered dispersion along unsteady flow. Al-Niami and Rushton [1] considered uniform flow whereas Kumar [11] took unsteady flow, against the dispersion in finite porous media. Most of these works have included the attenuation effect due to adsorption, first order radio-active decay and/or chemical reactions. Prakash [17] presented analytical solutions to predict temporal and spatial distribution of concentration in one-, two-, and three-dimensionally fully saturated uniform porous media flow for a point, line or parallelepiped source in an isotropic porous medium, R.R.Yadav and Dilip Kumar Jaiswal [18] discussed two-dimensional analytical solutions for point source contaminants transport.

One dimensional dispersion studies have been done along steady and unidirectional flow field. In almost all the solutions derived in one-dimension so for only longitudinal component of velocity where considered, neglecting vertical (transverse) component. The seepage velocity is exponentially decreasing function of time. Porous medium is considered homogeneous, isotropic saturated and of semi-infinite in horizontal plane. The seepage velocity is exponentially decreasing function of time. The first order decay term which is proportional to velocity is considered. Analytical solution is derived for uniform input source concentration with the help of Laplace transformation technique.

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Mathematical Model

Let pathogenic contaminant enters the groundwater from point source. The contaminant being of a considerably higher density than the groundwater moves towards the bottom of the thin aquifer along vertically downward, from its each point the contaminant is bound to spread in the horizontal plane along the unsteady. Let at one such point is the source concentration. The longitudinal and lateral directions in the horizontal plane extend up to infinity where no concentration at any time. The two-dimensional advection-dispersion equation is a partial differential equation describing hydrodynamic dispersion in homogenous, isotropic porous media is

$$\frac{\partial C}{\partial t} = D_x \frac{\partial^2 C}{\partial x^2} + D_y \frac{\partial^2 C}{\partial y^2} - u \frac{\partial C}{\partial x} - v \frac{\partial C}{\partial y} - \gamma C \qquad (1)$$

Where u is a horizontal flow velocity, v is a vertical flow velocity, D_x is dispersion coefficient along x-axis, D_y is dispersion coefficient along y-axis and γ is a first order decay. Let us take u_0 and v_0 are initial velocity components along x and y axes respectively.

$$u = u_0 e^{-mt}, v = v_0 e^{-mt} \quad and \quad \gamma = \gamma_0 e^{-mt} \qquad (2)$$

In (1) c(x, y, t) is the concentration at any time t in horizontal plane and γ_0 is the first order decay constant. Rumer [9] shows that, a relationship for unsteady and steady flow with a sinusoidal or exponentially varying flow velocity. So let α is the coefficient depends upon pore geometry and average pore size diameter of the porous medium.

$$D_x = \alpha u \quad and \quad D_y = \alpha v \qquad (3)$$

Using Eq. (3) in Eq. (2) we get.

$$D_x = D_{x_0} e^{-mt} \quad and \quad D_y = D_{y_0} e^{-mt} \qquad (4)$$

Where $D_{x_0} = \alpha u_0$ and $D_{y_0} = \alpha v_0$ are the initial dispersion coefficient components along the two respective directions. For the above problem the Initial and boundary conditions are as follows,

Primarily the groundwater is contaminant free therefore the initial and boundary is

$$c = 0, \quad t = 0, \; x \geq 0 \; and \; y \geq 0 \qquad (5)$$

when concentration is continuous across the inlet boundary and there is no solute flux at end of both boundaries. So the initial and boundary condition is

$$c = c_0, \; t > 0, \; x = 0, and \; y = 0 \qquad (6)$$

And

$$\frac{\partial c}{\partial x} = 0, \quad \frac{\partial c}{\partial y} = 0, \; t \geq 0, \; x \to \infty \quad and \quad y \to \infty \qquad (7)$$

Substituting Equations (2) and (4) in the partial differential equation (1) becomes,

$$e^{mt} \frac{\partial C}{\partial t} = D_{x_0} \frac{\partial^2 C}{\partial x^2} + D_{y_0} \frac{\partial^2 C}{\partial y^2} - u_0 \frac{\partial C}{\partial x} - v_0 \frac{\partial C}{\partial y} - \gamma_0 C \qquad (8)$$

By Crank [19] we introducing the new time variable T by following transformation

$$T = \int_0^t e^{-mt} \, dt = \frac{1}{m} \left(1 - e^{-mt} \right) \qquad (9)$$

For an expression e^{-mt} which is taken such that

e^{-mt} =1 for m = 0 or t = 0,

The new time variable obtained from equation (9) fulfils the conditions

T = 0 for t = 0 and T = ∞ for m = 0.

The above condition ensures that the nature of the initial condition does not change in the new time variable domain.

Thus the equation (8) becomes

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$$\frac{\partial C}{\partial T} = D_{x_0}\frac{\partial^2 C}{\partial x^2} + D_{y_0}\frac{\partial^2 C}{\partial y^2} - u_0\frac{\partial C}{\partial x} - v_0\frac{\partial C}{\partial y} - \gamma_0 C \qquad (10)$$

The following new space variable is introduced

$$X = x + y\sqrt{\frac{D_{y_0}}{D_{x_0}}} \qquad (11)$$

Therefore the two dimensional advection dispersion equation is reduced to one dimensional advection dispersion equation i.e.,

$$\frac{\partial C}{\partial T} = D\frac{\partial^2 C}{\partial X^2} - W\frac{\partial C}{\partial x} - \gamma_0 C \qquad (12)$$

Where

$$D = D_{x_0}\left(1 + \frac{{D_{y_0}}^2}{{D_{x_0}}^2}\right) \quad and \quad W = u_0\left(u_0 + v_0\sqrt{\frac{D_{y_0}}{D_{x_0}}}\right)$$

Initially, saturated flow of Newtonian fluid of concentration, C = 0, takes place in the soil media. At t=0, the concentration of the upper surface is immediately changed to $C = C_0$. Thus, the suitable boundary conditions for the given model are

$$\left.\begin{array}{ll} C(X,0) = 0 & X \geq 0 \\ C(0,T) = C_0 & T \geq 0 \\ C(\infty,T) = 0 & T \geq 0 \end{array}\right\} \qquad (13)$$

The problem then is to describe the concentration as a function of z and t .Where the input condition is assumed at the origin and a second type or flux type homogeneous condition is assumed. C_0 is initial concentration. To reduce equation (3) to a more familiar form, we take

$$C(X,T) = \Gamma(X,T)Exp\left[\frac{WX}{2D} - \frac{W^2T}{4D} - \gamma_0 T\right] \qquad (14)$$

$$\frac{\partial \Gamma}{\partial T} = D\frac{\partial^2 \Gamma}{\partial X^2} \qquad (15)$$

Equation (12) can be transform to (15) and its initial and boundary conditions (13) transform to (16) , when we substitute equation (14) to (12) and (13) respectively.

$$\left.\begin{array}{ll} \Gamma(0,T) = C_0\,Exp\left[\frac{W^2T}{4D} + \gamma_0 T\right] & T \geq 0 \\ \Gamma(X,0) = 0 & X \geq 0 \\ \Gamma(\infty,T) = 0 & T \geq 0 \end{array}\right\} \qquad (16)$$

Equation (15) may be solved for a time dependent influx of the fluid at X = 0. The solution of equation (15) may be obtained readily by use of Duhamel's theorem (Carslaw and Jaeger, 1947).

If $\boldsymbol{C = F(x,y,z,t)}$ is the solution of the diffusion equation for semi-infinite media in which the initial concentration is zero and its surface is maintained at concentration unity, then the solution of the problem in which the surface is maintained at temperature $\phi(t)$is

$$C = \int_0^t \phi(\tau)\frac{\partial}{\partial t}F(x,y,z,t-\tau)d\tau$$

This theorem is used principally for heat conduction problems, but the above has been specialized to fit this specific case of interest.

When we Applying Laplace transform to equation (15) is reduced to an ordinary differential equation, that is,

$$\mathscr{L}\left[\frac{\partial \Gamma}{\partial T}\right] = \mathscr{L}\left[D\frac{\partial^2 \Gamma}{\partial X^2}\right]$$

$$\frac{\partial^2 \overline{\Gamma}}{\partial X^2} = \frac{p}{D}\overline{\Gamma} \tag{17}$$

The solution of the equation is $\overline{\Gamma} = Ae^{-qX} + Be^{qX}$ where, $q = \pm\sqrt{\frac{p}{D}}$ boundary condition at X = 0 requires that $A = \frac{c_0}{p - \frac{W^2}{4D}}$ and The boundary condition as $X \to \infty$ requires that B = 0 thus the particular solution of the Laplace transformed equation is

$$\overline{\Gamma} = \frac{c_0}{p - \frac{W^2}{4D}} \quad e^{-qX}$$

The inversion Laplace transforms of the above function is gives, the result

$$\Gamma = 1 - erf\left(\frac{X}{2\sqrt{DT}}\right) = \frac{2}{\sqrt{\pi}} \int_{\frac{z}{2\sqrt{DT}}}^{\infty} e^{-\eta^2} d\eta$$

Using Duhamel's theorem, the solution of the problem with initial concentration zero and the time dependent surface condition at z = 0 is

$$\Gamma = \int_0^T \phi(\tau) \frac{\partial}{\partial T}\left[\frac{2}{\sqrt{\pi}} \int_{\frac{z}{2\sqrt{D(T-\tau)}}}^{\infty} e^{-\eta^2} d\eta\right] d\tau$$

Since $e^{-\eta^2}$ is a continuous function, it is possible to differentiate under the integral, which gives

$$\frac{2}{\sqrt{\pi}} \frac{\partial}{\partial T} \int_{\frac{z}{2\sqrt{D(T-\tau)}}}^{\infty} e^{-\eta^2} d\eta = \frac{X}{2\sqrt{DT}(T-\tau)^{3/2}} Exp\left[\frac{-X^2}{4D(T-\tau)}\right]$$

The solution to the problem is

$$\Gamma = \frac{\mathrm{X}}{2\sqrt{\pi \mathrm{D}}} \int_0^T \phi(\tau)\, Exp\left[\frac{-X^2}{4D(X-\tau)}\right] \frac{d\tau}{(X-\tau)^{3/2}} \tag{18}$$

Putting $\mu = \frac{X}{2\sqrt{D(T-\tau)}}$ then the equation (18) can be written as

$$\Gamma = \frac{2}{\sqrt{\pi}} \int_{\frac{\mathrm{z}}{2\sqrt{\mathrm{DT}}}}^{\infty} \phi\left(T - \frac{X^2}{4D\,\mu^2}\right) e^{-\mu^2}\, \mathrm{d}\,\mu \tag{19}$$

Since $\phi(T) = C_0\, Exp\left(\frac{W^2 T}{4D} + \gamma_0 T\right)$ the particular solution of the problem may be written as

$$\Gamma(\mathrm{X}, \mathrm{T}) = \frac{2\mathrm{C}_0}{\sqrt{\pi}} Exp\left(\frac{W^2 \mathrm{T}}{4D} + \gamma_0 T\right)\left\{\int_0^{\infty} Exp\left(-\mu^2 - \frac{\varepsilon^2}{\mu^2}\right) d\mu - \int_0^{\alpha} Exp\left(-\mu^2 - \frac{\varepsilon^2}{\mu^2}\right) d\mu\right\} \tag{20}$$

Where, $\alpha = \frac{X}{2\sqrt{DT}}$ and $\varepsilon = \sqrt{\frac{W^2}{4D} + \frac{K_d(1-n)}{n} - \gamma_0}\ \left(\frac{X}{2\sqrt{D}}\right)$

Integrating the integral part of (20) and substituting in (12) we get the final solution of the equation (12) is

$$\frac{c}{c_0} = \frac{1}{2}\left[\mathrm{Exp}\left\{\frac{\left(\beta - \sqrt{\beta^2 + \gamma_0}\right)X}{\sqrt{D}}\right\}.\mathrm{erfc}\left\{\frac{X - \sqrt{W^2 + 4\gamma_0 DT}}{2\sqrt{DT}}\right\}\right] + \frac{1}{2}\left[\mathrm{Exp}\left\{\frac{\left(\beta + \sqrt{\beta^2 + \gamma_0}\right)X}{\sqrt{D}}\right\}.\mathrm{erfc}\left\{\frac{X - \sqrt{w^2 + 4\gamma_0 DT}}{2\sqrt{DT}}\right\}\right] \quad (21)$$

Where $\beta = \frac{W}{2\sqrt{D}}$, $X = x + y\sqrt{\frac{D_{y_0}}{D_{x_0}}}$, $T = \frac{1}{m}\left(1 - e^{-mt}\right)$ $D = D_{x_0}\left(1 + \frac{{D_{y_0}}^2}{{D_{x_0}}^2}\right)$, $W = u_0\left(u_0 + v_0\sqrt{\frac{D_{y_0}}{D_{x_0}}}\right)$

To illustrate the concentration distribution of the obtained analytical solution in two-dimensional homogeneous porous medium in semi-infinite domain, an example has been chosen in which the different variables are given numerical values, where longitudinal and lateral seepage velocity and dispersion coefficients. Figures 1 to 4 are drawn for solution with different time 1.5, 2.0, 2.5 and 3 (days) for unsteady with the values of $u_0 = 0.95\ m/day$, $v_0 = 0.95\ m/day$, $D_{x_0} = 1.05\ m^2/day$ $D_{y_0} = 0.105\ m^2/day$, $\gamma_0 = 0.04\ (1/day)$ $m = 0.1$ $(1/\ day)$

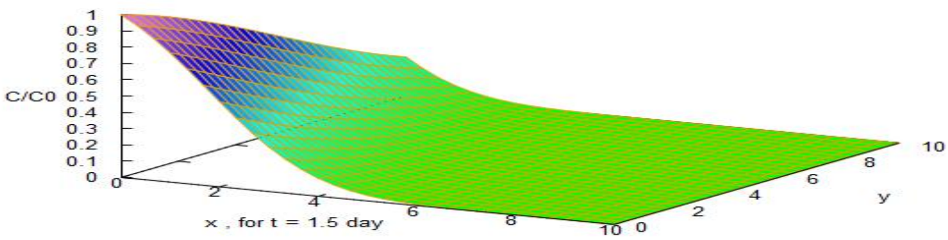

Fig. 1. Concentration profile for t = 1.5 day

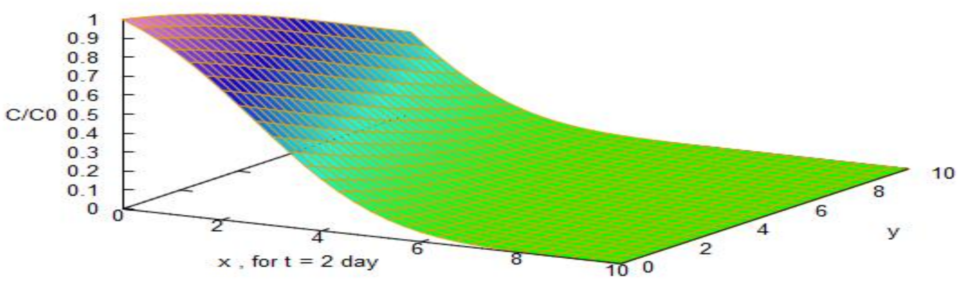

Fig. 2. Concentration profile for t = 2 day

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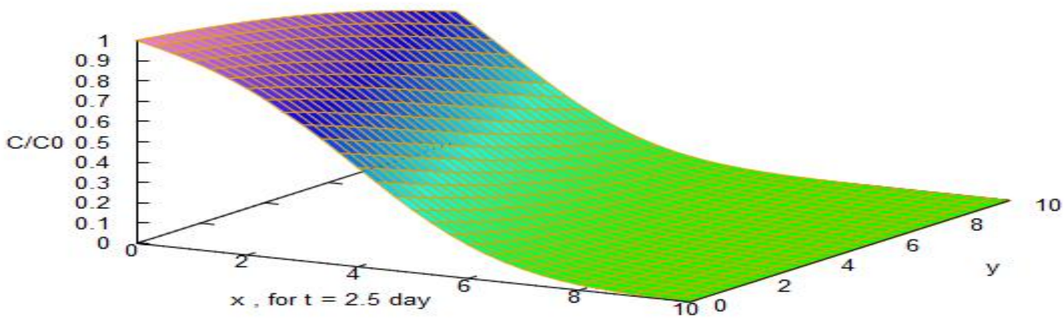

Fig. 3. Concentration profile for t = 2.5 day

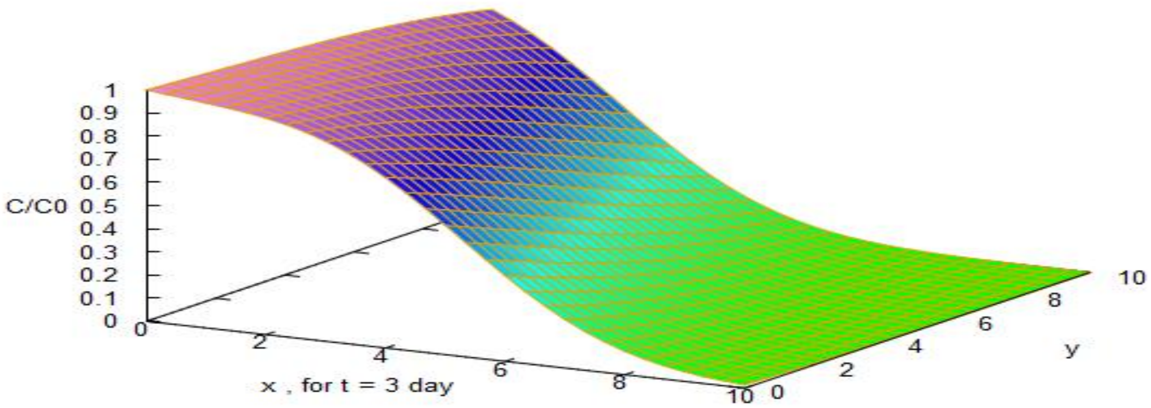

Fig. 4. Concentration profile for t = 3 day

RESULTS & DISCUSSIONS

The Concentration profile C/C_0 decreases with time. It means as the time increases dispersion process goes on dominating over the transport due to convection. In the case of unsteady flow similar but lesser effect would appear too. Though exponential form of seepage velocity is considered in the present analysis, there is no reason why other form of seepage velocity could not be used as long as the boundary conditions compatible. A horizontal two-dimensional transport model is selected because the groundwater flow is fundamentally horizontal and vertically mixing of the groundwater contaminant is a good approximation in such thin aquifer. The time dependent behaviour of contaminants in subsurface is of concern for many realistic problems where the concentration is observed or needs to be predicted at fixed positions. Problems of solute transport in two-dimension involving sequential first order decay reactions frequently arises in many groundwater systems.

This mathematical model is dependable with physical phenomenon of the longitudinal and lateral dispersion of contaminated in homogenous porous medium. From the above graphs we can conclude that concentration of the pathogenic contamination is decrease as the time increases. Laplace transform is a very useful method to get analytical solutions for solute transport in homogeneous porous media and under different flow atmosphere. The analytical expressions derived here are handy to the study of salinity intrusion in groundwater, useful in making quantitative predictions on the possible Contamination of groundwater supplies resulting from groundwater movement through buried wastes and other groundwater contamination problems.

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