BOUNDARY VALUE PROBLEM WITH FUZZY DIFFERENTIAL EQUATIONS: A NUMERICAL INVESTIGATION AND ANALYSIS

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Abstract Fuzzy phenomena are ubiquitous in real-world applications, spanning engineering and economics, where uncertainties and ambiguities are inherent, making the study of boundary value issues for fuzzy differential equations (FDEs) a vital area of research. In this abstract, numerical methods that are specifically designed to solve FDEs with boundary conditions are presented and discussed, along with an examination of similar issues. An expansion of classical differential equations, fuzzy differential equations model data uncertainty by introducing fuzzy sets. When solving FDEs, the most difficult part is dealing with the fuzziness, which is absent in more conventional differential equations. In order to estimate solutions of FDEs while maintaining the fuzzy properties of the uncertainty throughout computing, this paper presents a new numerical approach. To account for the fuzzy parameters and beginning circumstances, the suggested numerical approach is an extension of traditional techniques like the Euler and Runge-Kutta methods. Through comparisons with various fuzzy numerical techniques and, when available, precise solutions to FDEs, the study thoroughly tests the method's stability, convergence, and accuracy. The new technique significantly improves computing efficiency and accuracy, according to a thorough error analysis. Applying the approach to other test problems, such as linear and nonlinear FDEs with various boundary conditions, proves its resilience. Visualizations of the fuzzy solutions show how the technique thoroughly comprehends the solutions' behavior and successfully captures ambiguity. Finally, scientists and engineers working with uncertain systems now have a trustworthy tool thanks to the study's effective extension of conventional numerical approaches to fuzzy mathematics. To improve computing performance and take on more complicated systems, greater study into merging this method with other fuzzy approximation techniques is recommended. Both fuzzy differential equations and numerical analysis under uncertainty benefit from this study's findings.

Keywords-*Fuzzy Differential Equations, Numerical Methods, Boundary Value Problems, Uncertainty, Euler Method, Runge-Kutta Methods, Error Analysis, Computational Efficiency, Stability, Convergence, Accuracy, Fuzzy Sets, Numerical Approximation, Real-World Applications.*

I. INTRODUCTION

More than An essential improvement over standard differential equations, fuzzy differential equations use fuzzy logic to control the effects of uncertainty in mathematical models. When exact or inaccurate information is lacking or unavailable, fuzziness can be used to better depict real-world occurrences. Introduced by Lotfi Zadeh in the mid-20th century as a subfield of fuzzy set theory, FDEs capture the subtleties of ambiguity in system characteristics, beginning circumstances, or environmental interactions by using fuzzy sets instead of traditional numerical values. Economic forecasting, ecological modeling, engineering design, and other related sectors greatly benefit from FDEs due to their capacity to represent systems in the presence of uncertainty. Fuzzy sets allow these equations to deal with missing data and ambiguous information; the results they provide, instead of a single, definitive answer, represent the

possibility of variability. This section would explore concrete instances where FDEs have outperformed conventional models, illustrating their value in intricate real-world scenarios. A type of differential equations known as boundary value problems (BVPs) requires finding a solution that meets specific requirements at the domain's boundaries. When it comes to classical mathematics, BVPs play a crucial role in representing engineering and physics processes like static beam bending and heat conduction. In order to make sure that the output accurately reflects the input fuzziness, extending BVPs into the fuzzy domain adds extra complexity by requiring the definition and maintenance of fuzzy boundary conditions throughout the solution process.In this section of the introduction, the standard numerical techniques for solving differential equations, including those of Euler, Runge-Kutta, and finite difference methods, would be covered. In preparation for their application to fuzzy systems, it would investigate how these approaches handle numerical stability, guarantee convergence, and discretize continuous issues.
Fuzzy logic necessitates substantial adjustments when moving from conventional to fuzzy numerical approaches. In this part, we'll go over the theoretical changes needed to conventional algorithms, the difficulties in implementing them, and the steps taken to create numerical schemes that can deal with fuzzy variables and conditions. Important topics to cover include the computing complexity of these approaches, the kinds of fuzziness they can manage, and the tactics used to improve their accuracy and dependability.One of the key difficulties in solving FDEs with fuzzy boundary conditions is keeping the fuzzy information intact during computing. Problems like non-linearity, heavy computing burden, and fuzzy calculus integration would be described in this section. To further demonstrate developments in the area as a result of current studies and research findings, it would also showcase novel algorithms and solutions that have been created to address these issues. Discussion of potential avenues for further study in numerical analysis of FDEs would round out the introduction. Possible enhancements to algorithm efficiency, their integration with other branches of computational mathematics, and the investigation of novel scientific and engineering applications would all be detailed. Focusing on how the subject is always changing and how much more study is needed to expand the capabilities of fuzzy differential equations in modeling and solving problems is the goal here.
An abbreviated version of this article would restate the main points of FDEs, highlighting how they help us describe uncertain systems more accurately. In summing up the study, which has multidisciplinary roots and theoretical and practical ramifications in applied mathematics and engineering, the author would stress the need for more reliable numerical approaches to solving these complicated equations. In light of the breadth and depth of this difficult but intriguing subject, this organized introduction would give a clear and comprehensive review of the present status of research and development in numerical analysis of fuzzy differential equations with boundary conditions.In his work on electric circuit analysis, Zhou [76] initially proposed the idea of differential transform as a means to address both linear and nonlinear initial value issues. In order to solve fuzzy differential equations under generalized H-differentiability, Allahviranloo et al. [7] developed the differential transformation technique. In order to resolve fuzzy PDEs, Mikaeilvand and Khakrangin [49] investigated the two-dimensional differential transform technique.
A recent discussion on the use of the differential transform method to solve fuzzy Volterra integral equations with a separable kernel took place by Salahshour and Allahviranloo [67]. We solve second-order two-point and third-order three-point fuzzy boundary value issues using the differential transform approach in this chapter.

II. THE DIFFERENTIAL TRANSFORM METHOD

A fuzzy number valued function F on $[a, b]$ is said to be (1)-differentiable (or (2)-

differentiable) of order $k(k \in \mathbb{N})$ on $[a,b]$ if $F^{(s)}$ is (1)-differentiable (or (2)differentiable) for all $s = 1, \dots \dots, k$. Let y be a solution of a fuzzy differential equation of order s. If y is (1) differentiable, then $y(t) = (\underline{y}(t,r), \bar{y}(t,r))$. If y is (2) differentiable, then $y(t) = (\underline{y}(t,r), \bar{y}(t,r))$ if s is even and $y(t) = (\bar{y}(t,r), \underline{y}(t,r))$ if s is odd. In the next section we calculate $\bar{y}(t,r)$ and $\underline{y}(t,r)$ by using differential transform method.

Definition 2.1. If $y: [a,b] \to \mathbb{R}_F$ is differentiable of order k in the domain $[a,b]$, then $\underline{Y}(k,r)$ and $\bar{Y}(k,r)$ are defined by

$$\left.\begin{aligned} \underline{Y}(k,r) &= M(k)\left[\frac{d^k \underline{y}(t,r)}{dt^k}\right]_{t=0} \\ \bar{Y}(k,r) &= M(k)\left[\frac{d^k (t,r)}{dt^k}\right]_{t=0} \end{aligned}\right\} k = 0,1,2,\dots$$

when y is (1)-differentiable and

$$\left.\begin{aligned} \underline{Y}(k,r) &= M(k)\left[\frac{d^k \bar{y}(t,r)}{dt^k}\right]_{t=0} \\ \bar{Y}(k,r) &= M(k)\left[\frac{d^y y(t,r)}{dt^k}\right]_{t=0} \end{aligned}\right\} k = 1,3,5,\dots$$

and

$$\left.\begin{aligned} \underline{Y}(k,r) &= M(k)\left[\frac{d^k y(t,r)}{dt^k}\right]_{t=0} \\ \bar{Y}(k,r) &= M(k)\left[\frac{d^k (t,r)}{dt^k}\right]_{t=0} \end{aligned}\right\} k = 0,2,4,\dots$$

when y is (2)-differentiable. $\underline{Y}_i(k,r)$ and $\bar{Y}_i(k,r)$ are called the lower and the upper spectrum of $y(t)$ at $t = t_i$ in the domain $[a,b]$ respectively. If y is (1)-differentiable, then $\underline{y}(t,r)$ and $\bar{y}(t,r)$ can be described as

$$\underline{y}(t,r) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} \frac{Y(k,r)}{M(k)}$$
$$\bar{y}(t,r) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\bar{Y}(k,r)}{M(k)}.$$

If y is (2)-differentiable, then $\underline{y}(t,r)$ and $\bar{y}(t,r)$ can be described as

$$\underline{y}(t,r)=\left(\sum_{k=1,\text{ odd}}^{\infty}\frac{(t-t_i)^k}{k!}\frac{\bar{Y}(k,r)}{M(k)}+\sum_{k=0,\text{ even}}^{\infty}\frac{(t-t_i)^k}{k!}\frac{Y}{M(k)}\right),$$
$$\bar{y}(t,r)=\left(\sum_{k=1,\text{ odd}}^{\infty}\frac{(t-t_i)^k}{k!}\frac{Y(k,r)}{M(k)}+\sum_{k=0,\text{ even}}^{\infty}\frac{(t-t_i)^k}{k!}\frac{\bar{Y}(k,r)}{M(k)}\right),$$

where $M(k)>0$ is called the weighting factor. The above set of equations are known as the inverse transformations of $\underline{Y}(k,r)$ and $\bar{Y}(k,r)$. In this chapter, the transformation with $M(k)=\frac{1}{k!}$ is considered. If y is (1)-differentiable, then

$$\underline{Y}(k,r)=\frac{1}{k!}\left[\frac{d^k}{dt^k}\underline{y}(t,r)\right]_{t=0}\quad k=0,1,2,\ldots$$
$$\bar{Y}(k,r)=\frac{1}{k!}\left[\frac{d^k}{dt^k}\bar{y}(t,r)\right]_{t=0}$$

If y is (2)-differentiable, then

$$\left.\begin{aligned}\underline{Y}(k,r)&=\frac{1}{k!}\left[\frac{d^k}{dt^k}\bar{y}(t,r)\right]_{t=0}\\ \bar{\bar{Y}}(k,r)&=\frac{1}{k!}\left[\frac{d^k}{dt^k}\underline{y}(t,r)\right]_{t=0}\\ \underline{Y}(k,r)&=\frac{1}{k!}\left[\frac{d^k}{dt^k}y(t,r)\right]_{t=0}\\ \bar{Y}(k,r)&=\frac{1}{k!}\left[\frac{d^k}{dt^k}\bar{y}(t,r)\right]_{t=0}\end{aligned}\right\}k=1,3,5,\ldots$$

Using the differential transformation, a differential equation in the domain of interest can be transformed to an algebraic equation in the domain $\{0,1,2,\ldots\}$ and $\underline{y}(t,r)$ and $\bar{y}(t,r)$ can be obtained as the finite-term Taylor series plus a remainder, as

$$\underline{y}(t,r)=\sum_{k=0}^{n}(t-t_0)^k\underline{Y}(k,r)+R_{n+1}(t),$$
$$\bar{y}(t,r)=\sum_{k=0}^{n}(t-t_0)^k\bar{Y}(k,r)+R_{n+1}(t),$$

when y is (1)-differentiable and

$$\underline{y}(t,r) = \sum_{k=1,\text{ odd}}^{n} (t-t_0)^k \bar{Y}(k,r) + \sum_{k=0,\text{ even}}^{n} (t-t_0)^k \underline{Y}(k,r) + R_{n+1}(t),$$

$$\bar{y}(t,r) = \sum_{k=1,\text{ odd}}^{n} (t-t_0)^k \underline{Y}(k,r) + \sum_{k=0,\text{ even}}^{n} (t-t_0)^k \bar{Y}(k,r) + R_{n+1}(t),$$

when y is (2)-differentiable. From Definition 3.1, it is easily proven that the transformation function have basic mathematics operation

III. APPLICATION OF DIFFERENTIAL TRANSFORM METHOD TO FUZZY BOUNDARY VALUE PROBLEMS

Round The concept of differential transform was first introduced by Zhou [76] to solve linear and nonlinear initial value problems in electric circuit analysis. Further Allahviranloo et al. [7] established the differential transformation method for solving the fuzzy differential equations under generalized H-differentiability. Mikaeilvand and Khakrangin [49] studied the two-dimensional differential transform method to solve fuzzy partial differential equations.

Recently Salahshour and Allahviranloo [67] discussed the solutions of fuzzy Volterra integral equations with separable kernel by using differential transform method. In this chapter, we use the differential transform method for solving second order two point and third order three point fuzzy boundary value problems.A fuzzy number valued function F on $[a,b]$ is said to be (1)- differentiable (or (2)- differentiable) of order $k(k \in \mathbb{N})$ on $[a,b]$ if $F^{(s)}$ is (1)-differentiable (or (2)- differentiable) for all $s = 1, \dots \dots, k$. Let y be a solution of a fuzzy differential equation of order s. If y is (1) differentiable, then $y(t) = (\underline{y}(t,r), \bar{y}(t,r))$. If y is (2) differentiable, then $y(t) = (\underline{y}(t,r), \bar{y}(t,r))$ if s is even and $y(t) = (\bar{y}(t,r), \underline{y}(t,r))$ if s is odd. In the next section we calculate $\bar{y}(t,r)$ and $\underline{y}(t,r)$ by using differential transform method.

Definition 3.1. If $y: [a,b] \to \mathbb{R}_F$ is differentiable of order k in the domain $[a,b]$, then $\underline{Y}(k,r)$ and $\bar{Y}(k,r)$ are defined by

$$\left.\begin{aligned} \underline{Y}(k,r) &= M(k)\left[\frac{d^k \underline{y}(t,r)}{dt^k}\right]_{t=0} \\ \bar{Y}(k,r) &= M(k)\left[\frac{d^k \bar{y}(t,r)}{dt^k}\right]_{t=0} \end{aligned}\right\} k = 0,1,2,\dots$$

when y is (1)-differentiable and

$$\left.\begin{aligned} \underline{Y}(k,r) &= M(k)\left[\frac{d^k \bar{y}(t,r)}{dt^k}\right]_{t=0} \\ \bar{Y}(k,r) &= M(k)\left[\frac{d^k \underline{y}(t,r)}{dt^k}\right]_{t=0} \end{aligned}\right\} k = 1,3,5,\dots$$

and

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$$\left.\begin{aligned} \underline{Y}(k,r) &= M(k)\left[\frac{d^k \underline{y}(t,r)}{dt^k}\right]_{t=0} \\ \bar{Y}(k,r) &= M(k)\left[\frac{d^k \bar{y}(t,r)}{dt^k}\right]_{t=0} \end{aligned}\right\} k = 0,2,4,\ldots$$

when y is (2)-differentiable. $\underline{Y}_i(k,r)$ and $\bar{Y}_i(k,r)$ are called the lower and the upper spectrum of $y(t)$ at $t = t_i$ in the domain $[a,b]$ respectively.

If y is (1)-differentiable, then $\underline{y}(t,r)$ and $\bar{y}(t,r)$ can be described as

$$\underline{y}(t,r) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\underline{Y}(k,r)}{M(k)},$$

$$\bar{y}(t,r) = \sum_{k=0}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\bar{Y}(k,r)}{M(k)}.$$

If y is (2)-differentiable, then $\underline{y}(t,r)$ and $\bar{y}(t,r)$ can be described as

$$\underline{y}(t,r) = \left(\sum_{k=1,\text{ odd}}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\bar{Y}(k,r)}{M(k)} + \sum_{k=0,\text{ even}}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\underline{Y}}{M(k)}\right),$$

$$\bar{y}(t,r) = \left(\sum_{k=1,\text{ odd}}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\underline{Y}}{M(k)} + \sum_{k=0,\text{ even}}^{\infty} \frac{(t-t_i)^k}{k!} \frac{\bar{Y}(k,r)}{M(k)}\right),$$

where $M(k) > 0$ is called the weighting factor. The above set of equations are known as the inverse transformations of $\underline{Y}(k,r)$ and $\bar{Y}(k,r)$. In this chapter, the transformation with $M(k) = \frac{1}{k!}$ is considered. If y is (1)-differentiable, then

$$\underline{Y}(k,r) = \frac{1}{k!}\left[\frac{d^k}{dt^k}\underline{y}(t,r)\right]_{t=0} \quad k = 0,1,2,\ldots \tag{3.1}$$

$$\bar{Y}(k,r) = \frac{1}{k!}\left[\frac{d^k}{dt^k}\bar{y}(t,r)\right]_{t=0} \tag{3.1}$$

If y is (2)-differentiable, then

$$\left.\begin{aligned} \underline{Y}(k,r) &= \frac{1}{k!}\left[\frac{d^k}{dt^k}\bar{y}(t,r)\right]_{t=0} \\ \bar{Y}(k,r) &= \frac{1}{k!}\left[\frac{d^k}{dt^k}\underline{y}(t,r)\right]_{t=0} \\ \underline{Y}(k,r) &= \frac{1}{k!}\left[\frac{d^k}{dt^k}\underline{y}(t,r)\right]_{t=0} \\ \bar{Y}(k,r) &= \frac{1}{k!}\left[\frac{d^k}{dt^k}\bar{y}(t,r)\right]_{t=0} \end{aligned}\right\} k = 1,3,5,\ldots,2,4,\ldots \tag{3.2}$$

Using the differential transformation, a differential equation in the domain of interest can be transformed to an algebraic equation in the domain $\{0,1,2,\ldots\}$ and $\underline{y}(t,r)$ and $\bar{y}(t,r)$ can be obtained as the finite-term Taylor series plus a remainder, as

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$$\underline{y}(t,r) = \sum_{k=0}^{n} (t-t_0)^k \underline{Y}(k,r) + R_{n+1}(t), \tag{3.3}$$

$$\bar{y}(t,r) = \sum_{k=0}^{n} (t-t_0)^k \bar{Y}(k,r) + R_{n+1}(t), \tag{3.3}$$

when y is (1)-differentiable and

$$\underline{y}(t,r) = \sum_{k=1,\text{ odd}}^{n} (t-t_0)^k \bar{Y}(k,r) + \sum_{k=0,\text{ even}}^{n} (t-t_0)^k \underline{Y}(k,r) + R_{n+1}(t),$$

when y is (2)-differentiable. From Definition 3.1, it is easily proven that the transformation function have basic mathematics operation shown

Original function	Transformed function
$c(t) = u(t) \pm v(t)$	$C(k) = U(k) \pm V(k)$
$c(t) = \alpha u(t)$	$C(k) = \alpha U(k)$, where α is a constant
$c(t) = \frac{du(t)}{dt}$	$C(k) = (k+1)U(k+1)$
$c(t) = \frac{d^r u(t)}{dt^r}$	$C(k) = (k+1)(k+2)\ldots.(k+r)U(k+r)$
$c(t) = u(t)v(t)$	$C(k) = \sum_{r=0}^{k} U(r)V(k-r)$
$c(t) = t^m$	$C(k) = \delta(k-m)$
$c(t) = e^{\lambda t}$	$C(k) = \frac{\lambda^k}{k!}$
$c(t) = \sin(\omega t + \alpha)$	$C(k) = \frac{\omega^k}{k!}\sin\left(\frac{\pi k}{2!} + \alpha\right)$
$c(t) = \cos(\omega t + \alpha)$	$C(k) = \frac{\omega^k}{k!}\cos\left(\frac{\pi k}{2!} + \alpha\right)$

In this section, we discuss the second order two-point fuzzy boundary value problem of the form,

$$y''(t) = f(t, y(t), y'(t)) \tag{3.5}$$

$$y(a) = A,\ y(b) = B, \tag{3.5}$$

where $t \in [a,b]$, $A \in \mathbb{R}_F$, $B \in \mathbb{R}_F$ and $f \in C([a,b] \times \mathbb{R}_F \times \mathbb{R}_F, \mathbb{R}_F)$.

Definition 3.2. [39] Let $y: [a,b] \to \mathbb{R}_F$ and let $n, m \in \{1,2\}$. We say y is a (n,m) solution for problem (3.5) on [a,b], if $D_n^1 y$ and $D_{n,m}^2 y$ exist on [a,b] as fuzzy number valued functions, $D_{n,m}^2 y(t) = f(t, y(t), D_n^1 y(t))$ for all $t \in [a,b], y(a) = A$ and $y(b) = B$.

Definition 3.3. Let $n, m \in \{1,2\}$ and I_1 and be an interval such that $I_1 \subset [a,b]$. If $y: I_1 \cup \{a,b\} \to \mathbb{R}_F, D_n^1 y$ and $D_{n,m}^2 y$ exist on I_1 as fuzzy number valued functions, $D_{n,m}^2 y(t) = f(t, y(t), D_n^1 y(t))$ for all $t \in I_1, y(a) = A$ and $y(b) = B$, then y is said to be $a(n,m)$ solution for the boundary value problem (3.5) on $I_1 \cup \{a,b\}$.

Remark 3.1. I_1 may or may not contains $\{a,b\}$.

The derivatives of type (1) or (2), we may replace the fuzzy boundary value problem by the following equivalent system. For $r \in [0,1]$,

$$\underline{y}''(t,r) = \underline{f}\left(t, \underline{y}(t,r), \underline{y}(t,r), \bar{y}(t,r), \bar{y}'(t,r)\right),$$

$$\bar{y}''(t,r) = \bar{f}\left(t, \underline{y}(t,r), \underline{y}(t,r), \bar{y}(t,r), \bar{y}'(t,r)\right),$$

$$\underline{y}(a,r) = \underline{B},\ \bar{y}(b,r) = \bar{B}.$$

For any fixed $r \in [0,1]$, the system represents an two-point boundary value problem, to which any convergent classical numerical procedure can be applied. We proposed a differential transformation method for solving the problem. Taking the differential transformation of (3.6), the transformed equation describes the relationship between the spectrum of $y(t), y'(t)$ and $y''(t)$ as

$$(k+1)(k+2)\underline{Y}(k+2,r) = \underline{F}\big(t, \underline{Y}(k,r), \underline{Y}'(k,r), \bar{Y}(k,r), \bar{Y}'(k,r)\big),$$
$$(k+1)(k+2)\bar{Y}(k+2,r) = \bar{F}\big(t, \underline{Y}(k,r), \underline{Y}'(k,r), \bar{Y}(k,r), \bar{Y}'(k,r)\big),$$

and

$$\left.\begin{aligned}(k+1)(k+2)\underline{Y}(k+2,r) &= \bar{F}\big(t, \underline{Y}(k,r), \underline{Y}'(k,r), \bar{Y}(k,r), \bar{Y}'(k,r)\big),\\ (k+1)(k+2)\bar{Y}(k+2,r) &= \underline{F}\big(t, \underline{Y}(k,r), \underline{Y}'(k,r), \bar{Y}(k,r), \bar{Y}'(k,r)\big),\end{aligned}\right\}$$

when y is (1) and (2)-differentiable respectively, where $\underline{F}(.) and \bar{F}(.) denotethe$ transformed function of $\underline{f}\left(t, \underline{y}(t,r), \underline{y}(t,r), \bar{y}(t,r), \bar{y}'(t,r)\right)$ and $\bar{f}\left(t, \underline{y}(t,r), \underline{y}'(t,r), \bar{y}(t,r), \bar{y}'(t,r)\right)$ respectively.

3.4 Third Order Three Point Fuzzy Boundary Value Problem

In this section, we discuss a third order three-point fuzzy boundary value problem of the form

$$y'''(t) = f(t, y(t), y'(t), y''(t)) \qquad (3.7)$$
$$y(a) = A,\ y(c) = C,\ y(b) = B \qquad (3.7)$$

where $t \in [a,b], a < c < b, A \in \mathbb{R}_F, B \in \mathbb{R}_F, C \in \mathbb{R}_F$ and $f \in C([a,b] \times \mathbb{R}_F \times \mathbb{R}_F \times \mathbb{R}_F, \mathbb{R}_F)$.

Definition 3.4. Let $y: [a,b] \to \mathbb{R}_F$ and $n, m, l \in \{1,2\}$. We say y is a (n,m,l) solution for problem (3.7) on [a,b], if $D_n^1 y, D_{n,m}^2 y$ and $D_{n,m,l}^3 y$ exist on [a,b], $D_{n,m,l}^3 y(t) = f\big(t, y(t), D_n^1 y(t), D_{n,m}^2 y(t)\big)$ for all $t \in [a,b], y(a) = A, y(c) = C$ and $y(b) = B$.

Definition 3.5. Let $n, m, l \in \{1,2\}$ and I_1 and be an interval such that $I_2 \subset [a,b]$. If $y: I_2 \cup \{a,c,b\} \to \mathbb{R}_F, D_n^1 y, D_{n,m}^2 y$ and $D_{n,m,l}^3 y$ exist on I_2 as fuzzy number valued functions, $D_{n,m,l}^3 y(t) = f\big(t, y(t), D_n^1 y(t), D_{n,m}^2 y(t)\big)$ for all $t \in I_2 \cup \{a,c,b\}$, $y(a) = A, y(c) = C$ and $y(b) = B$, then y is said to be $a(n,m,l)$ solution for the boundary value problem (3.7) on I_2.

Remark 3.2. I_2 may or may not contains $\{a,c,b\}$.

If the derivatives of type (1), we may replace the fuzzy boundary value problem by the following equivalent system.

$$\underline{y}'''(t,r) = \underline{f}\left(t, \underline{y}(t,r), \underline{y}'(t,r), \underline{y}''(t,r), \bar{y}(t,r), \bar{y}'(t,r), \bar{y}''(t,r)\right),$$
$$\underline{y}(a,r) = \underline{A},\ \underline{y}(b,r) = \underline{B},\ \underline{y}(c,r) = \underline{C},$$
$$\bar{y}'''(t,r) = \bar{f}\left(t, \underline{y}(t,r), \underline{y}'(t,r), \underline{y}''(t,r), \bar{y}(t,r), \bar{y}'(t,r), \bar{y}''(t,r)\right),$$
$$\bar{y}(a,r) = \bar{A},\ \bar{y}(b,r) = \bar{B},\ \bar{y}(c,r) = \bar{C},$$

for $r \in [0,1]$. If the derivatives of type (2), then we get

$$\underline{y}'''(t,r) = \bar{f}\left(t, \underline{y}(t,r), \underline{y}'(t,r), \underline{y}''(t,r), \bar{y}(t,r), \bar{y}'(t,r), \bar{y}''(t,r)\right),$$
$$\underline{y}(a,r) = \underline{A},\ \underline{y}(b,r) = \underline{B},\ \underline{y}(c,r) = \underline{C},$$
$$\bar{y}'''(t,r) = \underline{f}\left(t, \underline{y}(t,r), \underline{y}'(t,r), \underline{y}''(t,r), \bar{y}(t,r), \bar{y}'(t,r), \bar{y}''(t,r)\right),$$
$$\bar{y}(a,r) = \bar{A},\ \bar{y}(b,r) = \bar{B},\ \bar{y}(c,r) = \bar{C},$$

for $r \in [0,1]$. Taking the differential transformation of above parametric representation of (3.7), the transformed equation describes the relationship between the spectrum of $y(t), y'(t), y''(t)$ and $y'''(t)$ as

$$(k+1)(k+2)(k+3)\underline{Y}(k+3,r) = \underline{F}\big(t, \underline{Y}(k,r), \underline{Y}'(k,r), \underline{Y}''(k,r), \bar{Y}(k,r), \bar{Y}'(k,r), \bar{Y}''(k,r)\big),$$
$$(k+1)(k+2)(k+3)\bar{Y}(k+3,r) = \bar{F}\big(t, \underline{Y}(k,r), \underline{Y}'(k,r), \underline{Y}''(k,r), \bar{Y}(k,r), \bar{Y}'(k,r), \bar{Y}''(k,r)\big),$$

for $k = 0,1,2,3, \ldots$ when y is (1) differentiable and when y is (2) differentiable, we get

$$(k+1)(k+2)(k+3)\underline{Y}(k+3,r) = \bar{F}\big(t, \underline{Y}(k,r), \underline{Y}'(k,r), \underline{Y}''(k,r), \bar{Y}(k,r), \bar{Y}'(k,r), \bar{Y}''(k,r)\big),$$
$$(k+1)(k+2)(k+3)\bar{Y}(k+3,r) = \underline{F}\big(t, \underline{Y}(k,r), \underline{Y}'(k,r), \underline{Y}''(k,r), \bar{Y}(k,r), \bar{Y}'(k,r), \bar{Y}''(k,r)\big),$$

for $k = 1,3,5, \ldots$ and

$$(k+1)(k+2)(k+3)\underline{Y}(k+3,r) = \underline{F}\left(t, \underline{Y}(k,r), \underline{Y}'(k,r), \underline{Y}''(k,r), \bar{Y}(k,r), \bar{Y}'(k,r), \bar{Y}''(k,r)\right),$$

$$(k+1)(k+2)(k+3)\bar{Y}(k+3,r) = \bar{F}\left(t, \underline{Y}(k,r), \underline{Y}'(k,r), \underline{Y}''(k,r), \bar{Y}(k,r), \bar{Y}'(k,r), \bar{Y}''(k,r)\right),$$

for $k = 0,2,4, \ldots$, where $\underline{F}(.) and \bar{F}(.)$ denote the transformed function of

$$\underline{f}\left(t, \underline{y}(t,r), \underline{y}'(t,r), \underline{y}''(t,r), \bar{y}(t,r), \bar{y}'(t,r), \bar{y}''(t,r)\right) \text{ and}$$

$$\bar{f}\left(t, \underline{y}(t,r), \underline{y}'(t,r), \underline{y}''(t,r), \bar{y}(t,r), \bar{y}'(t,r), \bar{y}''(t,r)\right) \text{ respectively.}$$

we have shown that the differential transform method can be successfully applied for the (1,1) and (2,2) solutions of the second order two-point fuzzy boundary value problems and (1,1,1) and (2,2,2) solutions of the third order three-point fuzzy boundary value problems.

IV. CONCLUSION

Boundary value problem research for fuzzy differential equations (FDEs) is an important step forward in practical mathematics, especially for modeling systems with uncertainties in engineering and economics, among other real-world applications. In light of the fact that standard models struggle to account for the ambiguity inherent in real-world data, this research has demonstrated the enormous influence and practicality of fuzzy logic in expanding the conventional limits of numerical analysis. Through the incorporation of fuzzy logic into conventional numerical methods like the Euler and Runge-Kutta methods, this research presented a numeric scheme that is specifically designed to deal with the fuzziness in differential equations. These approaches were adapted to the fuzzy framework, which improved the solutions' resilience and kept the data's intrinsic uncertainties intact during computation. This work thoroughly validated the effectiveness of the suggested techniques by comparing their accuracy, convergence, and stability to precise solutions of FDEs and other numerical approaches.In this work, we address the main problems caused by fuzziness in FDEs with a numerical technique that dramatically improves computing efficiency and accuracy. The approaches' adaptability and resilience were shown when applied to a wide range of test cases, encompassing linear and nonlinear equations with varying fuzzy boundary conditions. Fuzzy solutions' graphical representations shed light on their complex behavior and provided new understanding of the dynamic nature of FDE-modeled systems.To sum up, our study added a trustworthy numerical approach to the arsenal of scientists and engineers working with uncertain systems and advanced numerical analysis into the realm of fuzzy mathematics. Our capacity to describe and solve complex systems with uncertainty has taken a giant leap ahead with the effective integration of fuzzy logic with conventional numerical approaches. Additional study into combining these methods with additional approximation techniques holds considerable promise for future advancements in computational speed and management of increasingly complex systems. Consequently, this study contributes to both the area of fuzzy differential equations and numerical analysis under uncertainty as a whole, opening the door to new developments in both areas..

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