MANDELBROT GRAPH GENERATION VIA JULIA SETS

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ABSTRACT :
The goal of this paper is to connect the theory of fractal geometry to the theory of the much broader class graph theory using independence polynomial. We are particularly interested in Julia sets and Mandelbrot graphs of the complex polynomials of the form $f(x) = x^n + mx + r$ where $m, r \in \mathbb{C}$ and $n \geq 2$.This paper also focuses on the various Mandelbrot graph examples, as well as the connectivity of the Petersen graph as a special case.
Keywords: Graph; Independence Polynomial; Fractal; Julia set; Mandelbrot graph; Petersen graph.

INTRODUCTION :

The independence polynomial is introduced by I. Gutman and F. Harary in
1983 [3]. Let s_k denote the number of independent sets of size k , which are induced subgraphs of G , then $I(G,x)=\sum_{k=0}^{\alpha(G)} s_k x^k$ where $\alpha(G)$ is the independence number of G. The independence polynomials are almost everywhere, but it is an NP complete problem to determine the independence polynomial of a graph.

A fractal is typically a geometric shape that can be divided into parts, each of which is a smaller copy of the whole. This property is known as self-similarity. Fractals are often associated with chaotic dynamical systems.

PRELIMINARIES :

A graph's independence polynomial is a polynomial whose coefficients represent the number of independent sets in the graph. While computing this polynomial is straightforward for small values of n, it becomes challenging for larger graphs. Although the following recurrence relation is computationally inefficient, it can be used to compute I(G,z) for any graph G.

Theorem 1 [2]: Let G be a simple graph .Let $v \in V(G)$ and N[v] be the closed neighborhood of v. Then $I(G; x)=I(G - v; x) + xI(G - N[v]; x)$.

Definition 1 [3]: A Julia set is defined on the extended complex plane. The filled Julia set of a polynomial f is given by $K(f) = \{z \in \mathbb{C}: f^n(z) \nrightarrow \infty\}$.The Julia set J(f) is defined as the boundary of the filled Julia set, i.e.,$J(f) = \partial K(f)$.

Definition 2 [1]: The reduced independence polynomial of a graphG is defined as $R(G, z) = I(G, z) - 1$, since every independence polynomial has a constant term of 1.

JULIA SET OF REDUCED INDEPENDENCE POLYNOMIAL OF SOME GRAPHS :

Julia sets beautifully illustrate the diversity of complex analysis, as they are not simply a single mathematical object[5]. Typically, Julia sets are not smooth curves, making it challenging to compute them directly from their definition [4]

The connectivity of a fractal is determined solely by the number of vertices in the graph, remaining unaffected by the edges. The following theorems reveal the relationship between graphs and their connectivity.

Theorem 2 [1]: If G is a non-empty graph with independence number 2, having n vertices and m non-edges, then (i) $-\frac{n}{m} \leq Re(z) \leq 0$ and (ii)$Im(z) = 0$ unless n=3 , in which case $-\frac{\sqrt{3}}{2m} \leq Im(z) \leq \frac{\sqrt{3}}{2m}$.

Theorem 3 [1]: If G is a graph with independence number 2, having n = 4 vertices and m non-edges,

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then $J(R(G, z)) \subseteq [-\frac{4}{m}, 0]$.

Corollary 1 [1]: If G is a non-empty graph with independence number 2, having $n \geq 5$ vertices and m non-edges, then it lies outside the Mandelbrot graph.

Using these results, we can draw the following conclusions.

Proposition1: For Complete graph K_n, $J(R(K_n, z)) = J(nz) = \{0\}$.

Proof:

If a graph has independence number 1, it is a complete graph of order n for some $n \geq 2$. The reduced independence polynomial of K_n is nz. Since any non-zero point has an unbounded forward orbit, the union and limiting root set is $\{0\}$. Therefore, the result holds.

Proposition 2: Path graph on 3 vertices P_3,

$$J(R(P_3, z)) = J(z^2 + 3z) \subseteq [-3,0] \text{ X } \left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right].$$

Proof:

Using theorem 2, Julia set of path graph on three vertices contained in the box $[-3,0] \text{ X } \left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$ and is connected.

Proposition 3 : For Path graph on 4 vertices P_4, $J(R(P_4, z)) = J(3z^2 + 4z) \subseteq \left[-\frac{4}{3}, 0\right]$.

Proof:

$I(P4) = 3z^2 + 4z + 1$ and its reduced independence polynomial is $3z^2 + 4z$. So it has independence number 2, n =4vertices and m =3 non edges. Then from theorem 3, $J(R(P_4, z)) = J(3z^2 + 4z) \subseteq \left[-\frac{4}{3}, 0\right]$.

Proposition 4 : For Cycle graph on 4 vertices C_4, $J(R(C_4, z)) = J(2z^2 + 4z) \subseteq [-2,0]$.

Proof:

The graph $G=C_4$ has independence polynomial $2z^2+4z+1$ and has independence number 2,n=4,m=2.Using theorem 3, its reduced independence polynomial $\subseteq[-2,0]$.

CLASSIFICATION OF MANDELBROT GRAPHS :

Julia sets are fractals in nature. A fractal is being linked to a graph G.

Definition 3[1] : A graph G is called a Mandelbrot graph if J(I(G,z)) is connected. Mandelbrot graph is useful for the connectivity of a Julia set of independence polynomial. We denote M={G: G is a Mandelbrot graph}.

Using the preceding propositions, we can derive the following corollary.

Corollary 2: Complete graph of order n, $K_n \in M$.

Corollary 3: If G is a non empty graph with independence number m having n vertices denoted by $G_{m,n}$, then we have the following results.

$G_{2,2} \in M$ (ii) $G_{2,3} \in M$ (iii) $G_{2,4} \in M$ (iv) $G_{2,n} \notin M$, where $n \geq 5$.

TYPES OF MANDELBROT GRAPHS :

(i) BARBELL GRAPH

A Barbell graph Bar_n is a graph on 2n vertices formed by joining two copies of the complete graph

K_n with a single edge, known as a bridge. We denote this graph by Bar_n.

Fig.1 Barbell graph of different orders

Independence polynomial of Barbell graph of order n is given by a second degree polynomial in z. $I(Bar_n,z) = z^2(n^2 - 1) + 2nz + 1$ where n = 1,2,3,…

Critical points are found by solving the equation, $f'(z) = 2z(n^2 - 1) + 2n = 0$. Thus , $z_0 = \frac{n}{n-1}$, where $n \neq 1$ are the critical points of $I(Bar_n,z)$.The forward orbits of $I(Bar_n,z)$ at $z = \frac{n}{n-1}$ are $f^1(z_0) = \frac{1}{1-n^2}$,$f^2(z_0) = \frac{n(n-2)}{n^2-1}$, $f^3(z_0) = \frac{n^4-2n^3+n^2-1}{n^2-1}$-,…etc.

(1) If n=2, the forward orbits of independence polynomial of Barbell graph of order 2 at its critical point $z_0 = \frac{-2}{3}$ are given by $f^1(z_0) = \frac{-1}{3}$, $f^2(z_0)=0, f^3(z_0)=1$,..etc. So it is not bounded. Hence its Julia set is totally disconnected. Therefore it is not an element of Mandelbrot graph.
(2).The forward orbits of $I(Bar_n, z)$ at its critical points increase indefinitely, so all orbits are unbounded and its Julia set is totally disconnected. As a result, it is not an element of Mandelbrot graph.
Corollary 4: The independence polynomial of Barbell graph of order n where $n \neq 1$ is not an element of Mandelbrot graph.

(ii)COCKTAIL PARTY GRAPH

The Cocktail Party graph n is a graph on 2n vertices. The graph is formed by takingnpairsofverticessuchthattheverticesinanyonepairareadjacenttoboth vertices in any other pair. There is no edge between the two vertices within any given pair. We denote this graph by CP_n.

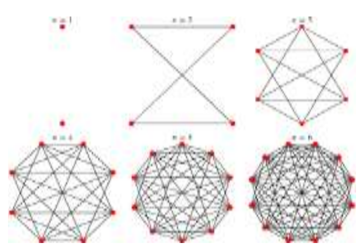


Fig.2 Cocktail Party graph of different orders

Independence polynomial of Cocktail Party graph of order n is given by
$I(CP_n, z) = nz^2 + 2nz + 1$ where $n = 1,2,3,\ldots$ Critical points are found by solving the equation $f'(z)=$

$1 + 2nz + nz^2$=0. Thus z_0=-1 is the critical point of I(CP_n,z).
The forward orbits of I(CP_n,z) at $z = -1$ are
$f^1(z_0) = 1 - n$, $f^2(z_0) = n^3 - 4n^2 + 3n + 1$,
$f^3(z_0) = n^7 - 8n^6 + 22n^5 - 20n^4 - 7n^3 + 12n^2 + 3n + 1$,…etc.

(1) If n=1, the forward orbits of independence polynomial of Cocktail Party graph of order 1 at its critical point $z_0 = -1$ are given by
$f^1(z_0) = 0$, $f^2(z_0)$=1, $f^3(z_0)$=4,…etc.So it is bounded. Hence its Julia set is totally disconnected. Therefore it is not an element of Mandelbrot graph.

(2) If n = 2, the forward orbits of independence polynomial of Cocktail Party graph of order 2 at its critical point $z_0 = -1$ are given by $f^1(z_0)$=-1, $f^2(z_0)$=-1, $f^3(z_0) = -1$,…etc. In general,when n=2,the sum of each forward orbits of independence polynomial of Cocktail Party graph of order 2 at its critical point z_0=-1 and 1 gives the root n=2. So the value of orbit is always -1 and its bounded. Hence its Julia set is connected. Therefore it is an element of Mandelbrot graph.

Hence we have the following result.

Corollary 5 : The forward orbit of I(CP_2,z) at its critical points are bounded and its Julia set is connected. As a result, it is the only Cocktail Party graph which an element of Mandelbrot graph.

Corollary 6: For all other values of n, the forward orbits of I(CP_n,z) at its critical points increase indefinitely, so all orbits are unbounded and its Julia set is totally disconnected. As a result, it is not an element of Mandelbrot graph.

COMPLETE BIPARTITE GRAPH :

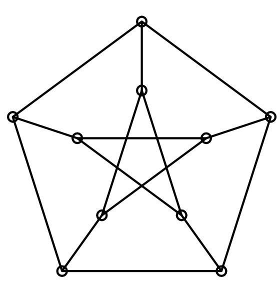
A complete bipartite graph is a bipartite graph (i.e., a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent) such that every pair of graph vertices in the two sets are adjacent. If there are p and q graph vertices in the two sets, the complete bipartite graph is denoted $K_{p,q}$.

Independence polynomial of complete bipartite graph of order n, $K_{n,n}$ is given by a nth degree polynomial in z. I($K_{n,n}$,z)= $2(1 + z)^n - 1$ .Critical points are found by solving the equation $f^1(z)=2n(1+z)^{n-1}=0$.Thus $z_0=-1,-1,-1,...(n-1)$ times is the critical point of I($K_{n,n}$,z).The forward orbits of I($K_{n,n}$, z)at $z = -1$ are $f^1(z_0)$=-1,, $f^2(z_0)$=1, $f^3(z_0)$=-1,…etc.

Corollary7: The forward orbits of I($K_{n,n}$,z)at its critical points are bounded for all values of n and its Julia set is connected. As a result, it is an element of Mandelbrot graph.

PETERSEN GRAPH :

The Petersen graph is an undirected graph with 10 vertices and 15 edges.It is a small graph that serves as a useful example and counter example for many problems in graph theory. It is denoted by $P_{5,2}$ and is 3



regular

Fig.3 Petersen graph $P_{5,2}$

The independence polynomial of Petersen graph is a fourth degree polynomial andis given by

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I**(P,z)**=$1+10z+30z^2+30z^3+5z^4$. Critical points are found by solving the equation $f'(z)=20z^3+90z^2+60z+10$.Thus $z_0 = -0.5, -0.268$ and -3.732 are the critical point of I(P,z).
The forward orbits of I(P,z) at $z = -0.5$ are $f^1(z_0)$=0.06, $f^2(z_0)$=1.71 , $f^3(z_0)$=298.58,…etc.
The forward orbits of I(P,z) at z=-0.268 are $f^1(z_0)$=-0.08, $f^2(z_0)$=0.38, $f^3(z_0)$=10.88,…etc.
The forward orbits of I(P,z) atz=-3.732 are $f^1(z_0)$=-207.92, $f^2(z_0)$=9076117389.96…etc.
The forward orbits of I(P,z) at its critical points increase indefinitely, so all orbits are unbounded and its Julia set is totally disconnected, As a result, it is not an element of Mandelbrot graph.
***Corollary*8**: The independence polynomial of Petersen graph is not an element of Mandelbrot graph.

CONCLUSION:
Summarizing the research pertaining to the graphs, the salient observations are listed as follows:
- Connectivity of a fractal does not depend on the connectivity of the graph.
- Julia set of independence polynomial of CP_2,$K_{n,n}$are all connected and therefore element of Mandelbrot graph.
- Petersen graph connectivity examined and found that its Julia set is disconnected.

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