FRACTIONAL CALCULUS OPERATOR AND ITS APPLICATIONS IN THE UNIVALENT FUNCATIONS

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Abstract: *The purpose of the present paper is to establish some results of Fractional Calculus Operator convex functions, In this paper, we obtain integral means inequalities for function and also we state integral means results for the classes studied as corollaries. For analytic functions g(z) and h(z) with g(0) = h(0), g(z) is said to be subordinate to h(z) if there exists an analytic function w(z) so that w(0) = 0,* $|w(z)| < 1 (z\epsilon U)$ *and g(z) = h(w(z)), we denote this subordination by* $g(z) \prec h(z)$*, Many papers in the theory of univalent functions are devoted to linear integral or integro-differential operators which map the class SS (of normalized analytic and univalent functions in the open unit disk U).*

Key-words : *univalent, starlike and convex functions; generalized fractional integrals and derivatives; Gauss and generalized hypergeometric functions; Meijer's G-functions and Fox's H-functions.*

1. Introduction and definitions

Let A denote the class of functions of the form (1.1)

$$f(z) = z + \sum_{n=2}^{\infty} a_n x^n$$

Which are analytic and univalent in the open disc

$u = z\{z : z \in C_1 || z | < 1\}$ Also denote by T the subclass of A consisting of functions of the form (1.2)

$$f(z) = \mathrm{z} - \sum_{n=2}^{\infty} \mathrm{a}_n z^n, a_n \geq 0, z \in u$$

Introduce and studied by Silverman.

Following Goodman, Ronning introduced and studied the following subclasses

(i) A function f ∈ A is said to be in the class k- uniformly starlike functions or order γ, if it satisfies the condition

$$Re\left\{\frac{zf^t(z)}{f\,(z)} - \gamma\right\} > k\left|\frac{zf^t(z)}{f\,(z)} - 1\right|, z \in u \qquad (1.3)\ 0 \leq$$

$$\gamma < 1\ and\ k \geq 0$$

(ii) A function $f \in A$ is said to be in the class UCV $(\gamma m\, k)$ k- uniformly convex functions or order γ, if it satisfies the condition

$$Re\left\{1 + \frac{f''(z)}{f'(z)} - \gamma\right\} > k\left|\frac{zf^{tt}(z)}{f^t(z)} - 1\right|, z \in u \qquad (1.4)$$

Indeed it follows form (1.3) and (1.4) that

$f\, \epsilon\, UCV\,(\gamma, k) \Leftrightarrow z\, f' \in Sp\,(\gamma, k)$ (1.5)

Definition 1.1.1: Given $\gamma\ (-1 \leq \gamma < 1), k\ (k \geq 0), k\ (k \geq o)$ and functions

$$\Phi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n\ and\ \Psi\,(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$$

Analytic in U, such that

$\lambda_n \geq 0, \mu_n \geq 0\ and\ \lambda_n \geq \mu_n\ for\ n \geq 2, we\ let\ f \in A\ is\ in\ U\,(\Phi, \Psi, \alpha, \beta) if\ (f^*\Psi)(z) \neq 0\ and$

$$Re\left\{\frac{(f*\Phi)(z)}{(f*\Psi)(z)}-\gamma\right\} \geq k\left|\frac{(f*\Phi)(z)}{(f*\Psi)(z)}-1\right|, \forall z \in u$$

Where (*) stands for the Hadamard product.
Further let $\mathrm{UT}(\Phi, \Psi, \alpha, \beta) = U(\Phi, \Psi, \alpha, \beta) \cap T$
We note that, by taking suitable choice of Φ, Ψ, α *and* β we obtain the following subclasses studied in literature.

1. $UT\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}, \gamma, 1\right) = TS_p(\gamma)$

2. $UT\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}, \gamma, k\right) = S_pT(\gamma, k)$

3. $UT\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, 0,1\right) = UCT$

4. $UT\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, 0, k\right) = UCT(k)$

5. $UT\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, \gamma, 1\right) = UCT(\gamma)$

6. $UT\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, \gamma, k\right) = UCT(\gamma, k)$

7. $UT\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}, \gamma, 0\right) = S_T*(\gamma)$

8. $UT\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}, \gamma, 0\right) = K_T(\gamma)$

9. $UT(\phi, \psi, \gamma, 0) = E_T(\phi, \psi, \gamma)$

10. $UT\left(\phi, \psi, \frac{1+\beta-2\alpha}{2(1-a)}, 0\right) = B_T(\phi, \psi, \alpha, \beta)$

In fact many subclasses of T are defined and studied to investigate coefficient estimates, extreme points, convolution properties and closure properties etc. suitably choosing ϕ, ψ, γ *and* k.

In this paper, we obtain integral means inequalities for function and also we state integral means results for the classes studied as corollaries. For analytic functions g(z) and h(z) with g(0) = h(0), g(z) is said to be subordinate to h(z) if there exists an analytic function w(z) so that w(0) = 0, $|w(z)| < 1(z\epsilon U)$ and g(z) = h(w(z)), we denote this subordination by $g(z) \prec h(z)$.
To prove our main result, we need the following lemmas.
Lemma 1.1.1: A function $f(z) \in UT(\phi, \psi, \gamma, k) or\ \gamma(-1 \leq \gamma < 1) and\ k(k \geq 0)$ if and only if (1.1.1)

$$\sum_{n=2}^{\infty}[(1+k)\lambda_n - (\gamma+k)\mu_n]\alpha_n \leq 1-\gamma$$

The result is sharp with the extremal functions

$$fn(z) = z - \frac{1-\gamma}{\sigma(\gamma, k, n)} z^n, n \geq 2 \qquad (1.1.2)$$

Where $\sigma(\gamma, k, n) = (1+k)\lambda_n - (\gamma+k)\mu_n, \gamma(-1 \leq \gamma < 1), k(k \geq 0) and\ n \geq 2$.

Lemma 1.1.2: If the function f(z) and g(z) are analytic in U with g(z) $\prec f(z)$ then

$\int_0^{2\pi} |g(re^{i\theta})|\,\eta d\theta \leq \int_0^{2\pi} |f||re^{i\theta}|\eta d\theta \eta > 0, z = re^{i\theta}$ *and* $0 < r < 1$

Integral mean

Applying Lemma 1.1.1 and lemma 1.1.2, we prove the following theorem.

Theorem 1.1.1 Let $\eta > 0.$ if $f(z) \in UT(\phi,\psi,\gamma,k) - 1 \leq \gamma < 1, k \geq 0$ *and* $\{\sigma(\gamma,k,n)\}_{n=2}^{\infty}$

Is non-decreasing sequence, then for

$z = re^{i\theta}$ *and* $0 < r < 1$ *we have* (2.1)

$$\int_0^{2\pi} |f|(re^{i\theta})|\eta d\theta \leq \int_0^{2\pi} |f_2)(re^{i\theta})|\eta d\theta$$

Where $f_2(z) = z - \frac{1-\gamma}{\sigma(\gamma,k,2)} z^2$

Proof. Let f(z) of the form (1.2) and $f_2(z) = z - \frac{1-\gamma}{\sigma(\gamma,k,2)} z^2$

Then we must show that

$$\int_0^{2\pi} |1 - \sum_{n=2}^{\infty} a_n z^{n-1}|\,\eta d\theta \leq \int_0^{2\pi} \left|1 - \frac{(1-\gamma)}{\sigma(\gamma,k,2)} z\right| \eta d\theta.$$

By Lemma 1.1.2 it suffices to show that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{1-\gamma}{\sigma(\gamma,k,2)^a}$$

Setting (2.2)

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1-\gamma}{\sigma(\gamma,k,2)} w(z)$$

From (2.2) and (1.1), we obtain.

$$|w(z)| = \left|\sum_{n=2}^{\infty} \frac{\sigma(\gamma,k,2)}{1-\gamma} a_n z^{n-1}\right|$$

$$\leq |z| \sum_{n=2}^{\infty} \frac{\sigma(\gamma,k,2)}{1-\gamma} a_n$$

$$\leq |z| < 1$$

This completed the proof of the Theorem 1.1.1

By taking different choices of ϕ, ψ, γ and k in the above theorem, we can state the following integral means results for various subclasses.

Corollary 1.2.2: Let $\eta > 0.$ *if* $f(z) \epsilon UT\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z^2)}, 0,1\right) = UCT$, then for $z = re^{i\theta}; 0 < r < 1$,we have (2.3)

ISSN: 2278-4632
Vol-14, Issue-6, No.01, June: 2024

$$\int_0^{2\pi} |f(re^{i\theta})|\eta d\theta \leq \int_0^{2\pi} |g_2(re^{i\theta})|\eta d\theta$$

Where $g_2(z) = z - \frac{z^2}{6}$

Corollary 1.2.3: Let

$\eta > 0. if f(z) \epsilon UT\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z^2)}, 0, k\right) = UCT(k)$

And $\geq 0, then\ for\ z = re^{i\theta}; 0 < r < 1, we\ have$.

$$\int_0^{2\pi} |f(re^{i\theta})|\eta d\theta \leq \int_0^{2\pi} |g_2(re^{i\theta})|\eta d\theta$$

Where $g_2(z) = z - \frac{z^2}{2(k+2)}$

2. Fractional Calculus

Many essentially equivalent definitions of fractional calculus (that is fractional derivatives and fractional integrals) have been given in the literature. We find it to be convenient to recall here the following definitions which are used earlier by Owa.

Definition 2.2.1. The fractional integral of order ξ is defined, for a function

$$f(z), by\ D_z^{-\xi} f(z) = \frac{1}{\Gamma(\xi)} \int_0^z \frac{f(\xi)}{(z-\varsigma)^{1-\xi}} d\varsigma \qquad (\xi > 0) \qquad (3.1)$$

Where the function $f(z)$ is analytic in a simply connected region of the z-plane containing the origin and the multiplicity of the function$(z-\varsigma)^{\xi-1}$ is removed by requiring the function log $(z-\varsigma)$ to be real when $z - \varsigma > 0$.

Definition 2.2.2. The fractional derivative of order ξ is defined, for a function $f(z)$, by

$$D_z \xi f(z) = \frac{1}{\Gamma(-\xi)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\varsigma)^{1-\xi}} d\varsigma \qquad (0 \leq \xi < 1) \qquad (3.2)$$

Where the function $f(z)$ is constrained, and the multiplicity of the function $(z-\varsigma)^{-\xi}$ is removed as in Definition 2.2.1.

Definition 2.2.3. Under the hypotheses of Definition 2.2.2, the fractional derivative of order $n + \lambda$ is defined by

$$D_z^{m+\xi} f(z) = \frac{d^m}{dz^m} D_z \xi f(z) \qquad (0 \leq \xi < 1; m \epsilon N_0) \qquad (3.3)$$

Conclusion: From Definition 2.2.2, we have $D_z^0 f(z) = f(z)$, which in view of Definition 2.2.3

yield

$$D_z^{m+0} f(z) = \frac{d^m}{dz^m} D_z^0 f(z) = f^{(m)}(z)$$

Thus, $\lim_{\xi \to 0} D_z^{-\xi} f(z) = f(z)$ and $\lim_{\xi \to 0} D_z^{1-\xi} f(z) = f'(z)$ we need the following definition of fractional integral operator given by Srivastava, Saigo and Owa.

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