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AN ANALYTICAL SOLUTION OF DIFFERENTIAL EQUATIONS OF TRANSVERSE DISPERSION IN UNSATURATED POROUS MEDIA WITH RETARDATION FACTOR

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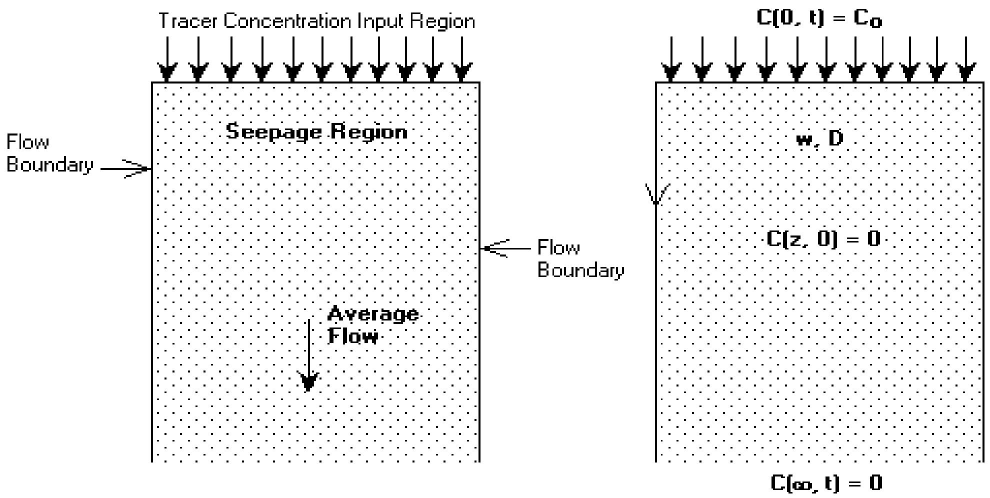
We have studied the mathematical models for transport of pollutants in transverse dispersion of unsaturated porous media one-dimensional advection-dispersion equation with spatially variable factor is derived using a generalized integral transform investigate the transport of sorbing but otherwise non-reacting solutes in hydraulic homogenous but geochemically heterogeneous formations. The solution is derived under conditions flow and arbitrary initial and inlet boundary conditions. The results obtained by this solution agree well with the results obtained by numerically inverting Laplace transform-generated solutions previously published in the literature. The solution is developed for a third or flux type inlet boundary condition, which is applicable when considering resident solute concentrations and a semi-infinite porous medium.

Physical Layout of the Model



The Advection-Dispersion equation along with initial condition and boundary conditions can be written as

$$\frac{\partial C}{\partial t}+\frac{1-n}{n}\frac{\partial S}{\partial t}=D\frac{\partial^2 C}{\partial z^2}-w\frac{\partial C}{\partial z}-\lambda C$$

The equilibrium isotherm between solution and adsorbed phase is given by

$$\frac{\partial S}{\partial t}=K_d\frac{\partial C}{\partial t}$$ K_d is the distribution coefficient

$$\frac{\partial C}{\partial t}+\frac{1-n}{n}K_d\frac{\partial C}{\partial t}=D\frac{\partial^2 C}{\partial z^2}-w\frac{\partial C}{\partial z}-\lambda C \qquad R\frac{\partial C}{\partial t}=D\frac{\partial^2 C}{\partial z^2}-w\frac{\partial C}{\partial z}-\lambda C$$

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$$\left[1+\frac{1-n}{n}K_d\right]\frac{\partial C}{\partial t}=D\frac{\partial^2 C}{\partial z^2}-w\frac{\partial C}{\partial z}-\lambda C \qquad \frac{\partial C}{\partial t}=D_1\frac{\partial^2 C}{\partial z^2}-w_1\frac{\partial C}{\partial z}-\lambda_1 C$$

Where

$$D_1=\frac{D}{R}, w_1=\frac{w}{R}, \lambda_1=\frac{\lambda}{R} \qquad R=1+\frac{1-n}{n}K_d$$ R is the Retardation factor

Thus the appropriate boundary conditions for the given model is

$$\left.\begin{array}{ll} C(z,0)=0 & z\geq 0 \\ C(0,t)=C_0e^{\gamma t} & t\geq 0 \\ C(\infty,t)=0 & t\geq 0 \end{array}\right\}$$

The problem then is to characterize the concentration as a function of z and t, where the input condition is assumed at the origin and a second type or flux type homogeneous condition is assumed. To reduce the governing equation into a Fick's law (diffusion equation), we have considered the moving coordinates

$$C(z,\ t)=\Gamma(z,\ t)\,Exp\left[\frac{w_1 z}{2D_1}-\frac{w_1{}^2 t}{4D_1}-\lambda_1 t\right]$$

The Advection-dispersion equations is of the familiar form

$$\frac{\partial \Gamma}{\partial t}=D_1\frac{\partial^2 \Gamma}{\partial z^2}$$

The initial and boundary condition transform to

$$\left.\begin{array}{ll} \Gamma(0,\ t)=C_0\ Exp\left[\frac{w_1{}^2}{4D_1}t+(\lambda_1-\gamma)t\right] & t\geq 0 \\ \Gamma(z,\ 0)=0 & z\geq 0 \\ \Gamma(\infty,\ t)=0 & t\geq 0 \end{array}\right\}$$

If C=F(x, y, z, t) s the solution of the diffusion equation for semi-infinite media in which the initial concentration is zero and its surface is maintained at concentration unity, then the solution of the problem in which the surface is maintained at temperature f(t) is (Duhamel's Theorem)

$$C=\int_0^t \phi(\lambda)\frac{\partial}{\partial t}F(x,y,z,t-\lambda)\,d\lambda$$

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This theorem is used principally for heat conduction problems, but the above has been specialized to fit this specific case of interest. consider now the problem in which intial concentration is zero and boundary is maintained at concentration unity. The boundary conditions are

$$\left.\begin{array}{ll}\Gamma(0,\,t)=0 & t\geq 0\\ \Gamma(z,\,0)=1 & z\geq 0\\ \Gamma(\infty,\,t)=0 & t\geq 0\end{array}\right\}$$

The Laplace Transform of the equation is given by

$$L\left[\frac{\partial\Gamma}{\partial t}\right]=D_1\frac{\partial^2\Gamma}{\partial z^2}$$

Hence it is reduced into an ordinary differential equation

$$\frac{d^2\overline{\Gamma}}{dz^2}=\frac{p}{D_1}\overline{\Gamma}$$

The solution of the equation is $\overline{\Gamma}=Ae^{-qz}+Be^{qz}$ where $q=\pm\sqrt{\frac{p}{D_1}}$

After substitution, solution of the equation is given by

$$\frac{C}{C_0}=\frac{1}{2}Exp\left[\frac{w_1z}{2D_1}-\gamma t\right]\left[Exp\left[\sqrt{\frac{w_1^{\,2}+4D_1(\lambda_1-\gamma)}{2D_1}}z\right]\cdot erfc\left[\frac{z+\sqrt{w_1^{\,2}+4D_1(\lambda_1-\gamma)}}{2\sqrt{D_1t}}t\right]+\right.$$

$$\left.\left[Exp\left[-\frac{w_1^{\,2}+4D_1(\lambda_1-\gamma)}{2D_1}z\right].erfc\left[\frac{z+\sqrt{w_1^{\,2}+4D_1(\lambda_1-\gamma)}}{2\sqrt{D_1t}}t\right]\right]$$

Where the boundaries are symmetrical, then the solution of the problem is given by first term of the equation. The second term due to asymmetric boundary imposed in the more general problem. However , it should be noted that, if a point at great distance away from the source is considered then it is possible to approximate the boundary conditions by $C(-\infty, t) = C_0$ which lead to a symmetrical solutions.

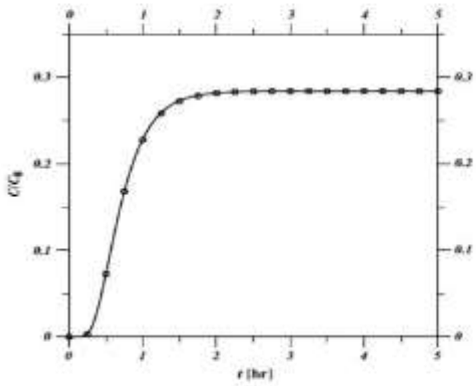

Fig. 1: Break-through-curve for C/C_0 v/s time

for z=10m, R=1.0, λ=0.5 & γ = 0

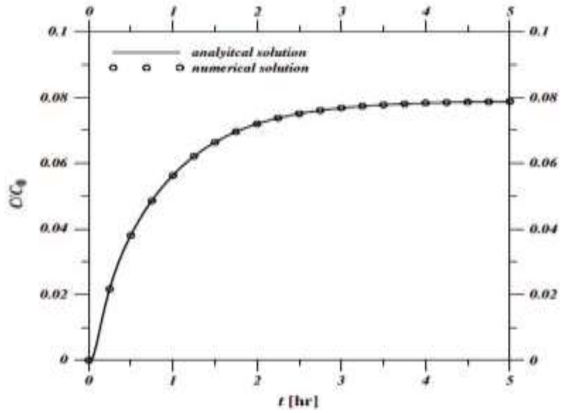

Fig. 2: Break-through-curve for C/C_0 v/s time

for z=10m, R=1.0, λ=0.5 & γ = 0.25

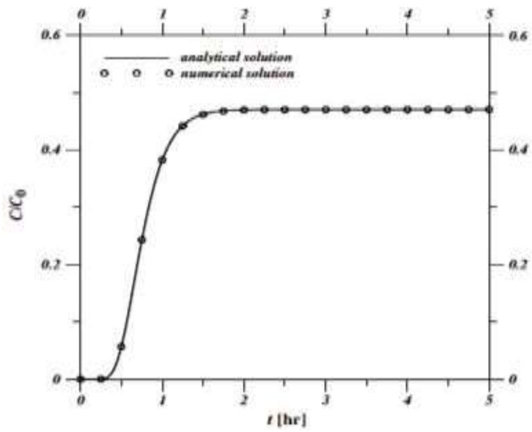

Fig. 3: Break-through-curve for C/C_0 v/s time

for z=10m, R=1.0, λ=0.5 & γ = 0.5

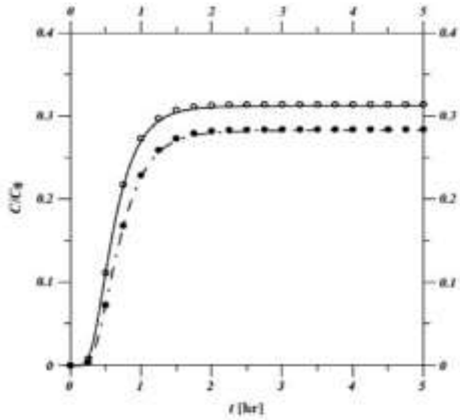

Fig. 4: Break-through-curve for C/C_0 v/s time

for z=10m, R=1.0, λ=0.5, γ = 0.75 & 1.0

Result and Discussions

This study presents analytical solutions for one-dimensional advection–dispersion equations in unsaturated porous medium in finite domain. The transform method coupled with the generalized integral transform technique is used to obtain the analytical solutions. Solutions are obtained for both first- and third-type inlet boundary conditions. The developed analytical solutions for finite domain are compared with solutions for the semi-infinite domain to clarify how the exit boundary influences the one-dimensional transport in a porous medium system.

The main limitations of the analytical methods are that the applicability is for relatively simple problems. The geometry of the problem should be regular. The properties of the soil in the region considered must be homogeneous in the sub region. The analytical method is somewhat more flexible than the standard form of other methods for one-dimensional transport model. Figures 1 to 4 represents the concentration profiles verses time in the adsorbing media for depth z = 10m and Retardation factor R=1. It is seen that for a fixed velocity w, dispersion coefficient D and distribution

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coefficient K_d, C/C_0 decreases with depth as porosity n decreases due to the distributive coefficient K_d and if time increases the concentration decreases for different time and decay chain.

Accordingly, the analytical solutions derived for the finite domain will thus be particularly useful for analyzing the one-dimensional transport in unsaturated porous medium with a large dispersion coefficient whereas the analytical solution for semi-infinite domain is recommended to be applied for a medium system with a small dispersion coefficient. Moreover, the developed solution is especially useful for validating numerical model simulated solution because realistic problems generally have a finite domain.

From this paper, we conclude that the mathematical solutions have been developed for predicting the possible concentration of a given dissolved substance in steady unidirectional seepage flows through semi-infinite, homogeneous, and isotropic porous media subject to source concentration that vary exponentially with time for spatially variable retardation factor using a change of variable and integral transform technique. The expressions take into account the contaminants as well as mass transfer from the liquid to the solid phase due to adsorption. For simultaneous dispersion and adsorption of a solute, the dispersion system is considered to be adsorbing at a rate proportional to its concentration.

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