Fuzzy Anti Inner Product Space and its Properties

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Abstract: In the present paper we have studied some properties of fuzzy anti inner product space and defined level completeness of fuzzy anti inner product space. We have introduced the notion of α-fuzzy orthogonality and fuzzy orthogonality and its properties. The relationship between α-fuzzy orthogonality and fuzzy orthogonality has been established. Fuzzy anti Hilbert space has also been introduced and its projection theorem has been given.

Keywords: α fuzzy orthogonality, fuzzy orthogonality, fuzzy anti Hilbert space, projection theorem.

1. Introduction: The concept of fuzzy set was first introduced by Zadeh in 1965 [10] which provides a natural framework for generalization of many concepts of general topology to fuzzy topology. Later Katraras [5] and Felbin [3] introduced the notion of fuzzy norm on a linear space in different ways. Several other workers have studied various properties of fuzzy normed linear spaces [1, 2, 8]. The definition of fuzzy inner product function on a linear space was introduced by Majumdar and Samanta [7] and they also studied some properties of fuzzy inner product function. They induced a fuzzy norm from the fuzzy inner product function and derived many important results related to minimizing vector, fuzzy orthogonality and parallelogram law.

In 2019 Sinha [9] introduced the definition of fuzzy anti inner product space and proved parallelogram law and polarization identity on fuzzy anti inner product space.

In this paper we have introduced the notion of fuzzy orthogonality. We have defined fuzzy anti- Hilbert space and relationship between α-fuzzy orthogonality and fuzzy orthogonality has been established.

2. Preliminaries: In this section some definitions and preliminary results have been given which will be used later in this paper.

Definition 2.1 [9]: Let X be a linear space over the field C of complex number. Let $\mu^* : X \times X \times C \to [0,1]$ be a mapping such that the following holds,

1. (FaIP1) For $s,t \in C, \mu^*(x+y,z,|t|+|s|) \leq \max\{\mu^*(x,z,|t|), \mu^*(y,z,|s|)\}$,

2. (FaIP2) For $s,t \in C, \mu^*(x,y,|st|) \le \max\{\mu^*(x,x,|s|^2), \mu^*(y,y,|t|^2)\}$,
3. (FaIP3) For $t \in C, \mu^*(x,y,t) = \mu^*(y,x,\bar{t})$,
4. (FaIP4) $\mu^*(\alpha x,y,t) = \mu^*\left(x,y,\frac{t}{|\alpha|}\right), \alpha(\ne 0) \in C, t \in C$
5. (FaIP5) $\mu^*(x,x,t) = 1, \forall t \in C/R^+$,
6. (FaIP6) $\mu^*(x,x,t) = 0, \forall t > 0$ if and only if $x = 0$
7. (FaIP7) $\mu^*(x,x,\bullet): R \to [0,1]$ is a monotonic non increasing function of R and $\lim_{t\to\infty} \mu^*(x,x,t) = 0$.

Then μ^* is said to be fuzzy anti inner product (FaIP in short) function on X and the pair (X, μ^*) is called a fuzzy anti inner product space.

Theorem 2.1 [9]: Let (X, μ^*) is a fuzzy anti inner product space. Then the function $N^*: X \times R \to [0,1]$ defined by

$$N^*(x,t) = \mu^*(x,x,t^2), \forall t \in R \text{ and } t > 0 \qquad \text{.......... (1)}$$

$$= 1, \quad \forall t \in R \text{ and } t \le 0$$

is a fuzzy anti norm induced by the FaIP. Now if μ^* satisfying the following condition :

(FaIP8) $(\mu^*(x,x,t^2) < 1, \forall t > 0) \Rightarrow x = 0$ and

(FaIP9) For all $x, y \in X$ and $p, q \in R$,

$$\mu^*(x+y,x+y,2q^2) \wedge \mu^*(x-y,x-y,2p^2) \le \mu^*(x,x,p^2) \wedge \mu^*(y,y,q^2)$$

Then $\|x\|_\alpha = \wedge\{t > 0 : N^*(x,t) < \alpha, \alpha \in (0,1)\}$ is a decreasing family of norms (α-norm) on X, satisfying parallelogram law. Then using polarization identity we can get ordinary inner product, called the α-inner product as

$$\langle x,y \rangle_\alpha = X_\alpha + iY_\alpha, \ \alpha \in (0,1) \qquad \text{------------(2)}$$

Where $X_\alpha = \frac{1}{4}\left(\|x+y\|_\alpha^2 - \|x-y\|_\alpha^2\right)$ and $Y_\alpha = \frac{1}{4}\left(\|x+iy\|_\alpha^2 - \|x-iy\|_\alpha^2\right)$

Definition 2.2 [4]: Let (X, N^*) be fuzzy anti normed linear space and $\alpha \in (0,1)$. A sequence $\{x_n\}$ in X is said to be α-convergent in X if there exists $x \in X$ such that $\lim_{n\to\infty} N^*(x_n - x, t) < 1-\alpha$, $\forall\ t > 0$ and x is called the limit of $\{x_n\}$.

Definition 2.3 [4]: Let (X, N^*) be fuzzy anti normed linear space and $\alpha \in (0,1)$. A sequence $\{x_n\}$in X is said to be α-Cauchy in X if $\lim_{n\to\infty} N^*(x_n - x_{n+p}, t) < 1-\alpha$, $\forall t > 0$ and $p = 1,2,3.....$.

Definition 2.4 [4]: Let (X, N^*) be fuzzy anti normed linear space and $\alpha \in (0,1)$. It is said to be α-complete if any α-Cauchy sequence in X, α-converges to a point in X

3. Fuzzy Orthogonality, α-Fuzzy Orthogonality and Fuzzy anti Hilbert space:

In this section we have established some results regarding fuzzy anti-inner product function and introduced the notion of α-fuzzy orthogonality and fuzzy orthogonality. We have also established a relationship between α-fuzzy orthogonality and fuzzy orthogonality. We have also proved projection theorem on fuzzy anti- Hilbert space.

Theorem 3.1 : Let(X, N^*) be a fuzzy anti normed linear space. Assume that $x, y \in X$ and $s, t \in C$ and max $\{N^*(x,|st|), N^*(y,|st|)\} \leq$ max $\{N^*(x,|s|^2), N^*(y,|t|^2)\}$.

Define $\mu^* : X \times X \times C \to [0,1]$ as $\mu^*(x, y, s+t) = 1$ if $x = y$ and $s+t \in C - R^+$, elsewhere as $\mu^*(x, y, s+t) = N^*(x,|s|) \wedge N^*(y,|t|)$ then μ^* is a fuzzy anti inner product on X .

Proof : (FaIP1) For $s,t \in C$ and $x, y, z \in X$ we have,

$$\mu^*(x+y, z, |s|+|t|) = \mu^*(x+y, z, |s|+|t|+0)$$

$$= N^*(x+y, |s|+|t|) \wedge N^*(z, 0)$$

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$$\leq \max\{N^*(x,|s|), N^*(y,|t|)\} \wedge N^*(z,0)$$

$$\leq \max\{N^*(x,|s|) \wedge N^*(z,0), N^*(y,|t|) \wedge N^*(z,0)\}$$

So $\mu^*(x+y, z, |s|+|t|) \leq \max\{\mu^*(x,z,|s|), \mu^*(y,z,|t|)\}$

(FaIP2) $\mu^*(x,y,|st|) = \mu^*(x,y,|st|+0)$

$$= N^*(x,|st|) \wedge N^*(y,0)$$

$$= N^*(x,|st|) \quad \text{as } N^*(y,0) = 1$$

Similarly $\mu^*(x,y,|st|) = N^*(y,|st|)$

Thus $\mu^*(x,y,|st|) = N^*(x,|st|) = N^*(y,|st|)$ ………. (1)

$$= \max\{N^*(x,|st|), N^*(y,|st|)\}$$

$$\leq \max\{N^*(x,|s|^2), N^*(y,|t|^2)\} \quad \text{by (1)}$$

$$= \max\{\mu^*(x,x,|s|^2), \mu^*(y,y,|t|^2)\}$$

(FaIP3) $\mu^*(x,y,t) = \mu^*(x,y,t+0)$

$$= N^*(x,|t|) \wedge N^*(y,0)$$

$$= N^*(x,|t|) \quad \text{as } N^*(y,0) = 1$$

So, $\mu^*(x,y,\bar{t}) = N^*(y,|\bar{t}|) = \mu^*(y,x,\bar{t})$

(FaIP4) $\mu^*(\alpha x, y, t) = N^*(\alpha x, |t|) = N^*\left(x, \frac{|t|}{|\alpha|}\right) = \mu^*\left(x, y, \frac{t}{|\alpha|}\right)$

(FaIP5) $\mu^*(x,x,t) = 1, \forall t \in C - R^+$ (by definition)

(FaIP6) $\mu^*(x,x,t) = 0, \forall t > 0$

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$\Leftrightarrow N^*(x,t)=0, \forall t>0$

$\Leftrightarrow x=0$

(FaIP7) Since $\mu^*(x,x,.)=N^*(x,.)$ and $N^*(x,.)$ is a monotonic non-increasing function of R and $\lim_{t\to\infty} N^*(x,t)=0$ .Thus μ^* is a fuzzy anti inner product on X . □

Theorem 3.2: Let (X,μ^*) be a fuzzy anti inner product space satisfying (FaIP8), (FaIP9) and $\langle,\rangle_\alpha^*$ be its α -inner product, $\alpha\in(0,1)$. Define $\mu_1^*: X\times X\times C\to[0,1]$ as $\mu_1^*(x,y,t)=1$ if $x=y$ and $\forall t\in C-R^+$, elsewhere as $\mu_1^*(x,y,t)=\wedge\{\alpha\in(0,1):\left|\langle x,y\rangle_\alpha^*\right|<t\}$ then μ_1^* is a fuzzy anti inner product on X if $\left|\langle,\rangle_\alpha^*\right|$ is decreasing function of R .

Proof : (FaIP1) For $s,t\in C$ and $x,y,z\in X$ we have to show that

$\mu_1^*(x+y,z,|s|+|t|)\le\max\{\mu_1^*(x,z,|s|),\mu_1^*(y,z,|t|)\}$

Let $p=\mu_1^*(x,z,|s|)$ and $q=\mu_1^*(y,z,|t|)$, without loss of generality assume that $p\ge q$. Let $p\ge q$ then $\exists\alpha$ and β such that $\left|\langle x,z\rangle_\alpha^*\right|<|s|$ and $\left|\langle y,z\rangle_\beta^*\right|<|t|$. Let $\gamma=\alpha\vee\beta$ thus $\left|\langle x,z\rangle_\gamma^*\right|<\left|\langle x,z\rangle_\alpha^*\right|<|s|$ and $\left|\langle y,z\rangle_\gamma^*\right|<\left|\langle y,z\rangle_\beta^*\right|<|t|$. As $\langle,\rangle_\alpha$ is decreasing.

Now, $\left|\langle x+y,z\rangle_\gamma^*\right|=\left|\langle x,z\rangle_\gamma^*+\langle y,z\rangle_\gamma^*\right|\le\left|\langle x,z\rangle_\gamma^*\right|+\left|\langle y,z\rangle_\gamma^*\right|<|s|+|t|$

Therefore $\mu_1^*(x+y,z,|s|+|t|)\le\gamma$, thus $\mu_1^*(x+y,z,|s|+|t|)\le\max\{\mu_1^*(x,z,|s|),\mu_1^*(y,z,|t|)\}$

(FaIP2) For $s,t\in C$ and $x,y\in X$ we have to show that

$\mu_1^*(x,y,|st|)\le\max\{\mu_1^*(x,x,|s|^2),\mu_1^*(y,y,|t|^2)\}$.Let $p=\mu_1^*(x,x,|s|^2)$ and $q=\mu_1^*(y,y,|t|^2)$ without loss of generality assume that $p\ge q$. So, $\exists\alpha$ such that $\left|\langle x,x\rangle_\alpha^*\right|<|s|^2$ and $\exists\beta$ such that $\left|\langle y,y\rangle_\beta^*\right|<|t|^2$. Let $\gamma=\alpha\vee\beta$ thus $\left|\langle x,x\rangle_\gamma^*\right|<\left|\langle x,x\rangle_\alpha^*\right|<|s|^2$ and $\left|\langle y,y\rangle_\gamma^*\right|<\left|\langle y,y\rangle_\beta^*\right|<|t|^2$. (Since $\left|\langle,\rangle_\alpha^*\right|$ is decreasing)

Therefore, $\left|\langle x,y\rangle_r^*\right|^2 \le \left|\langle x,x\rangle_r^*\langle y,y\rangle_r^*\right| < |s|^2|t|^2 \Rightarrow \left|\langle x,y\rangle_r^*\right| < |st|$

$\Rightarrow \mu_1^*(x,y,|s|) \le \max\{p,q\}$. So, $\mu_1^*(x,y,|st|) \le \max\left\{\mu_1^*\left(x,x,|s|^2\right), \mu_1^*\left(y,y,|t|^2\right)\right\}$.

(FaIP3) For $t \in C$, $\mu_1^*(x,y,t) = \mu_1^*(x,y,\bar{t}) = 1$ if $x = y$ and $\forall t \in C - R^+$. Let $t \in C$ and $x \neq y$ then $\mu_1^*(x,y,t) = \wedge\left\{\alpha \in (0,1) : \left|\langle x,y\rangle_\alpha^*\right| < |t|\right\}$

$$= \wedge\left\{\alpha \in (0,1) : \left|\langle x,y\rangle_\alpha^*\right| < |\bar{t}|\right\} = \mu_1^*(x,y,\bar{t})$$

(FaIP4) For $c \in C$, $\mu_1^*(cx,y,t) = \wedge\left\{\alpha \in (0,1) : \left|\langle cx,y\rangle_\alpha^*\right| < |t|\right\}$

$$= \wedge\left\{\alpha \in (0,1) : |c|\left|\langle x,y\rangle_\alpha^*\right| < |t|\right\}$$

$$= \wedge\left\{\alpha \in (0,1) : \left|\langle x,y\rangle_\alpha^*\right| < \frac{|t|}{|c|}\right\} = \mu_1^*\left(x,y,\frac{t}{|c|}\right)$$

(FaIP5) $\mu_1^*(x,x,t) = 1, \forall t \in C - R^+$ (by definition)

(FaIP6) $\mu_1^*(x,x,t) = 0, \forall t > 0$

$$\Leftrightarrow \wedge\left\{\alpha \in (0,1) : \left|\langle x,x\rangle_\alpha^*\right| < t\right\} = 0$$

$$\Leftrightarrow \langle x,x\rangle_\alpha^* = 0, \forall \alpha \in (0,1)$$

$$\Leftrightarrow x = 0$$

(FaIP7) $\mu_1^*(x,x,t) = \wedge\left\{\alpha \in (0,1) : \left|\langle x,x\rangle_\alpha^*\right| < |t|\right\}$

$$= \wedge\left\{\alpha \in (0,1) : \|x\|_\alpha^{*2} < t\right\}, \forall t > 0$$

$$= \wedge\left\{\alpha \in (0,1) : \|x\|_\alpha^* < \sqrt{t}\right\}$$

Consider if, $t_1 < t_2 \Leftrightarrow \sqrt{t_1} < \sqrt{t_2}$

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$$\Leftrightarrow \{\alpha \in (0,1) : \|x\|_\alpha^* < \sqrt{t_1}\} \subset \{\alpha \in (0,1) : \|x\|_\alpha^* < \sqrt{t_2}\}$$

$$\Leftrightarrow \wedge\{\alpha \in (0,1) : \|x\|_\alpha^* < \sqrt{t_1}\} \geq \wedge\{\alpha \in (0,1) : \|x\|_\alpha^* < \sqrt{t_2}\}$$

$$\Leftrightarrow \mu_1^*(x,x,t_1) \geq \mu_1^*(x,x,t_2)$$

Since $\mu_1^*(x,x,.) : R^+ \to [0,1]$ is decreasing function of R and $\lim_{t\to\infty} \mu^*(x,x,t) = 0$. Thus μ_1^* is a fuzzy anti inner product on X. □

Definition 3.1: Let (X, μ^*) is a fuzzy anti inner product space satisfying (FaIP8) then X is said to be level complete if for any $\alpha \in (0,1)$, every Cauchy sequence converges in X with respect to the α-norm, $\| \ \|_\alpha$, generated by the fuzzy anti norm N^* which is induced by fuzzy anti inner product μ^*.

Theorem 3.3 : Let (X, μ^*) is a fuzzy anti inner product space satisfying (FaIP8), (FaIP9) and $M(\neq \phi)$ be a convex subset of X which is level complete. Let $x \in X$ then for each $\alpha \in (0,1)$, $\exists$ a unique $y_0^\alpha \in M$ such that $m_{y_0^\alpha}^{(\alpha)} = \inf_{y \in M}\left\{m_y^{(\alpha)}\right\}$, where $m_{y_0^\alpha}^{(\alpha)} = \inf\{t \in R^+ : N^*(y_0^\alpha, t) < \alpha\}$, N^* being the fuzzy anti norm induced by the FaIP μ^*.

Proof: Let for each $\alpha \in (0,1)$ and $y \in M$, then $m_y^\alpha = \inf\{t \in R^+ : N^*(y,t) < \alpha\} = \|y\|_\alpha$ where $\| \ \|_\alpha$ is the α-norm induced from the fuzzy anti norm N^* which is obtained from the fuzzy anti inner product μ^*. By (2) of theorem 2.1, $(X, \langle,\rangle_\alpha)$ is an inner product space for each $\alpha \in (0,1)$ as M is level complete and convex. So for each $\alpha \in (0,1)$, M is a convex complete subset of $(X, \langle,\rangle_\alpha)$. Hence by minimizing vector theorem in crisp inner product space we get the result. □

Definition 3.2 : Let $\alpha \in (0,1)$ and (X, μ^*) be a fuzzy anti inner product space satisfying (FaIP8), (FaIP9). If $x, y \in X$ be such that $\langle x, y\rangle_\alpha = 0$, then we say that x, y are α-fuzzy orthogonal to each other and it is denoted by $x \perp_\alpha y$. Let M be a subset of X and $x \in X$. If

$\langle x, y\rangle_\alpha = 0, \forall y \in M$ then we say that x is α-fuzzy orthogonal to M and is denoted by $x \perp_\alpha M$.

Definition 3.3 : Let (X, μ^*) be a fuzzy anti inner product space satisfying (FaIP8), (FaIP9). If $x, y \in X$ be such that $\langle x, y\rangle_\alpha = 0$, then we say that x, y are fuzzy orthogonal to each other and it is denoted by $x \perp_\alpha y$.

Thus $x \perp y$ if and only if $x \perp_\alpha y, \forall \alpha \in (0,1)$.

Theorem 3.4: Let (X, μ^*) be a fuzzy anti inner product space satisfying (FaIP8), (FaIP9) such that $\mu^*(x, x, \bullet)$ is strictly decreasing and lower semi continuous for any $x \in X$. Then for $x, y \in X$, $x \perp y$ if and only if $\mu^*(x+y, x+y, t^2) = \mu^*(x-y, x-y, t^2)$ and $\mu^*(x+iy, x+iy, t^2) = \mu^*(x-iy, x-iy, t^2)$, t > 0.

Proof: Let $x \perp y$ then $\langle x, y\rangle_\alpha = X_\alpha + iY_\alpha = 0$, $\alpha \in (0,1)$

Where $X_\alpha = \frac{1}{4}\left(\|x+y\|_\alpha^2 - \|x-y\|_\alpha^2\right)$ and $Y_\alpha = \frac{1}{4}\left(\|x+iy\|_\alpha^2 - \|x-iy\|_\alpha^2\right)$, $\alpha \in (0,1)$

Now, $\|x+y\|_\alpha^2 - \|x-y\|_\alpha^2 = 0, \forall \alpha \in (0,1)$ ..... (1)

and $\|x+iy\|_\alpha^2 - \|x-iy\|_\alpha^2 = 0, \forall \alpha \in (0,1)$ ..... (2)

from (1) we get $\|x+y\|_\alpha^2 = \|x-y\|_\alpha^2, \forall \alpha \in (0,1)$ .....(3)

$\Rightarrow \inf\{t > 0 : N^*(x+y, t) < \alpha\} = \inf\{s > 0 : N^*(x-y, s) < \alpha\}$, $\forall \alpha \in (0,1)$

$\Rightarrow \inf\{t > 0 : \mu^*(x+y, x+y, t^2) < \alpha\} = \inf\{s > 0 : \mu^*(x-y, x-y, s^2) < \alpha\}$, $\forall \alpha \in (0,1)$

Now if possible let, $\mu^*(x+y, x+y, s^2) \neq \mu^*(x-y, x-y, s^2)$, for some $s > 0$.

Let us assume that $\mu^*(x+y, x+y, s^2) < \mu^*(x-y, x-y, s^2)$ and let $\mu^*(x-y, x-y, s^2) = \alpha_0$

Then by our assumption that $\mu^*(x,x,\bullet)$ is strictly decreasing and lower semi continuous for all $x \in X$, we get

$$\inf\{t>0:\mu^*(x+y,x+y,t^2)<\alpha_0\}>\inf\{r>0:\mu^*(x-y,x-y,r^2)<\alpha_0\}$$

$\Rightarrow \|x+y\|^*_{\alpha_0} > \|x-y\|^*_{\alpha_0}$ which is contradiction of (3).

So, $\mu^*(x+y,x+y,t^2)=\mu^*(x-y,x-y,t^2), \ \forall t>0$

Similarly we can prove that $\mu^*(x+iy,x+iy,t^2)=\mu^*(x-iy,x-iy,t^2), \forall t>0$. □

Theorem 3.5: Let (X,μ^*) be a fuzzy anti inner product space satisfying (FaIP8), (FaIP9). If $\mu^*(x,x,\bullet)$ is strictly decreasing and continuous for all $x \in X$ then $x \perp_\alpha y$ iff $\{\mu^*(x+y,x+y,t^2)<\alpha$ iff $\mu^*(x-y,x-y,t^2)<\alpha, \forall t>0\}$ and $\{\mu^*(x+iy,x+iy,t^2)<\alpha$ iff $\mu^*(x-iy,x-iy,t^2)<\alpha, \forall t>0\}$.

Proof: Let $x \perp_\alpha y$ then $\langle x,y\rangle_\alpha = X_\alpha + iY_\alpha = 0$, $\alpha \in (0,1)$

$\Rightarrow X_\alpha = 0$

$\Rightarrow \|x+y\|^2_\alpha - \|x-y\|^2_\alpha = 0$ ….. (1)

and $Y_\alpha = 0$

$\Rightarrow \|x+iy\|^2_\alpha - \|x-iy\|^2_\alpha = 0$ ….. (2)

Now from (1) we have, $\|x+y\|^2_\alpha = \|x-y\|^2_\alpha$

So, $\inf\{t>0:\mu^*(x+y,x+y,t^2)<\alpha\}=\inf\{s>0:\mu^*(x-y,x-y,s^2)<\alpha\}$ (by theorem 3.4) , so $\mu^*(x+y,x+y,t^2)=\mu^*(x-y,x-y,s^2)$. □

Definition 3.4: Let (X,μ^*) be a fuzzy anti inner product space. X is said to be a fuzzy anti Hilbert space , if it is level complete.

Definition 3.5: Let (X, N^*) be a fuzzy anti normed linear space. A subset F is said to be level fuzzy closed (l-fuzzy closed) if for each $\alpha \in (0,1)$ and

for any sequence $\{x_n\}$ in F and $x \in X$, $\left(\overline{\lim_{n\to\infty}} N^*(x_n - x, t) < \alpha, \forall t > 0\right) \Rightarrow x \in F$.

Proposition 3.1 : Let (X, N^*) be a fuzzy anti normed linear space satisfying the following condition, $\left(N^*(x,t) < 1, \forall t > 0\right) \Rightarrow x = \underline{0}$ and $F \subset X$. Then F is l-fuzzy closed iff F is closed with respect to $\| \|_\alpha^*$ for each $\alpha \in (0,1)$.

Definition 3.6: Let $N^*(x,t)$ be a fuzzy anti normed linear space assume further that for all $t > 0$, $N^*(x,t) < 1 \Rightarrow x = 0$ then $\|x\|_\alpha = \inf\{t > 0 | N^*(x,t) < \alpha, \alpha \in (0,1)\}$ is a decreasing family of norms on X. Also if $F \subset X$ then F is l-fuzzy closed iff F is closed with respect to $\| \|_\alpha, \forall \alpha \in (0,1)$.

Definition 3.7: Let H be a fuzzy anti Hilbert space and let Y be l- fuzzy closed subspace of H. Then its α –fuzzy orthogonal complement is defined as

$$Y^{\perp_\alpha} = \{z \in H | Z \text{ is } \alpha\text{-fuzzy orthogonal to } Y\} = \{z \in H | \langle z, y\rangle_\alpha = 0, y \in Y\}$$

Theorem 3.6 (Projection Theorem): Let Y be any l-fuzzy closed subspace of a fuzzy anti Hilbert space H then $H = Y \oplus Y^{\perp_\alpha}$.

Proof: As H is a fuzzy anti Hilbert space. So for each $\forall \alpha \in (0,1)$, H is level complete with respect to $\| \|_\alpha$. As Y be any l- fuzzy closed so Y is closed with respect to $\| \|_\alpha, \forall \alpha \in (0,1)$. Thus for each $\forall \alpha \in (0,1)$, Y is closed subspace of a fuzzy anti Hilbert space $(H, \langle\ \rangle_\alpha)$. So for each $\forall \alpha \in (0,1)$, H can be expressed as $H = Y \oplus Y^{\perp_\alpha}$.

*On behalf of all authors, the corresponding author states that there is no conflict of interest.

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