Congruence's on Ternary Gamma Semi Rings-II

K. Praveen Kumar[1] P.Siva Prasad[2],D.Madhusudana Rao[3], Ch. Srimannarayana[4]

[1]Research Scholar, Department of Mathematics, AcharyaNagarjuna University, Nagarjuna Nagar, Guntur.

[2] Professor, Department of Sciences & Humanities, VFSTR Deemed to be University, Vadlamudi, Guntur (Dt.), A.P., India.Email:pusapatisivaprasad@gmail.com

[3]Professor, Department of Mathematics, VSR & NVR College, Tenali. Email:dmrmaths@gmail.com

[4]Associate Professor, Department of Mathematics, TJPS College, Guntur.

Abstract: The main theme of this paper is some of the concepts on "congruence ternary Gamma semirings","quasi commutative ternary ternary semirings","weakly commutative ternary semirings". On the other hand using the "permutable" property some results are obtained.This paper also contains some results on "weakly commutative ternary gamma semirings". One of the properties like "permutable", "seaparative" etc. has taken over a "weakly commutative ternary gamma semiring".

1. Introduction

Pondelicek[1] was first introduced the concept of "congruencies on semigroups".The study of "separating semigroups" begin with the famous Howitt[4] and Zuclevaman[4. In [9] D. M. Rao, et al introduced the notion of "Ternary Gamma semiring" in 2015. The main theme of this paper was to classify the "congruence's" on "ternary gamma semi rings" and to study the important properties of "congruence's" on "ternary gamma semi rings".

2. Preliminaries

Let T and Γ be "two non-empty sets" then T is said to be a "ternary Γ-semiring" if ∃a mapping from $T\times\Gamma\times T\times\Gamma\times T$ to T maps $(t_1,\chi,t_2,\phi,t_3)\rightarrow[t_1\chi t_2\phi t_3]$ satisfying the condition that(T, +) is a "commutative semigroup", (T, Γ , [ ]) is a "ternary Γ-semigroup",and "ternary multiplication" distributes over addition from three sides.

A "ternary Γ-semiring" T is a "***commutative ternary Γ-semiring***" if $a\alpha b\beta c = b\alpha c\beta a = c\alpha a\beta b = b\alpha a\beta c = c\alpha b\beta a = a\alpha c\beta b$ $\forall a,b,c\in$ T and $\alpha, \beta\in\Gamma$. In the set notation one can define "commutative ternary Γ-semiring" as$a\Gamma b\Gamma c = b\Gamma c\Gamma a = c\Gamma a\Gamma b = b\Gamma a\Gamma c = c\Gamma b\Gamma a = a\Gamma c\Gamma b \forall a, b, c\in$T.

A "ternary Γ-semiring" T is known as "***multiplicatively left(lateral, right) cancellative***" [MLC(MMC, MRC)] if $a\Gamma b\Gamma x = a\Gamma b\Gamma y$($a\Gamma x\Gamma b = a\Gamma y\Gamma b$, $x\Gamma a\Gamma b = y\Gamma a\Gamma b$) implies that $x = y$ for all $a, b, x, y\in$ T. If T is MLC, MRC and MMC, then T is known as "cancellative".

Theorem 2.1: A "ternary Γ-semiring" T is "multiplicatively cancellative" (MC) if any one of the following conditions holds :
(i) T is "multiplicatively left and right cancellative"
(ii) T is "multiplicatively laterally cancellative"
(in) For any $u\in$T, the "equation" $u\Gamma x\Gamma u = u\Gamma y\Gamma u$ implies that $x = y$.

If "ternary Γ-semiring T" is said to be "***quasi commutative ternary Γ-semiring***" provided for each a, b, $c\in$ T and $\gamma\in$ Γ, there exists a odd natural number n such that $a\gamma b\gamma c = (b\gamma)^n a\gamma c = b\gamma c\gamma a = (c\gamma)^n b\gamma a = c\gamma a\gamma b = (a\gamma)^n c\gamma b$. In the set notation we can define as for each a, b, $c\in$T , there exists a "odd natural number n" such that, $a\Gamma b\Gamma c = (b\Gamma)^n a\Gamma c = b\Gamma c\Gamma a = (c\Gamma)^n b\Gamma a = c\Gamma a\Gamma b = (a\Gamma)^n c\Gamma b$.

The set of all "natural numbers" under usual "addition and ternary multiplication" is a "quasi-commutative ternary Γ-semiring".

3. Structure of "Separative Ternary Γ-Semiring"

Definition 3.1: A "ternary Γ-semiring" (T, Γ, +, [ ]) is known as "***separative***" if $(x\Gamma)^2x= x\Gamma x\Gamma y$, $(y\Gamma)^2y= y\Gamma y\Gamma x \Rightarrow x = y$ and $(x\Gamma)^2x= y\Gamma y\Gamma x$, $(y\Gamma)^2y= x\Gamma x\Gamma y \Rightarrow x = y \ \forall x, y$ in T.

Theorem 3.2: Let (T, Γ, +, [ ]) be a "commutative ternary Γ-semiring". Define a relation ρ on a "ternary Γ-semiring" T by $x\rho y \Rightarrow x\Gamma(y\Gamma)^{n-1}y= (y\Gamma)^n y$, $y\Gamma(x\Gamma)^{n-1}x= (x\Gamma)^n x$, for any even +ve integer n and for any $x, y\in$ T, then ρ is a "separative congruence" on T.

Proof: Since $x\Gamma(x\Gamma)^{n-1}x= (x\Gamma)^n x= (x\Gamma)^{n-1}x\Gamma x \ \forall x$ in T $\Rightarrow x\rho x$. ρ is reflexive.

Let $x\rho y$ for some x, y in T $\Rightarrow x\Gamma(y\Gamma)^{n-1}y= (y\Gamma)^n y$, $y\Gamma(x\Gamma)^{n-1}x= (x\Gamma)^n x$.

Replace x by y, y by x we have $y\Gamma(x\Gamma)^{n-1}x= (x\Gamma)^n x$, $x\Gamma(y\Gamma)^{n-1}y= (y\Gamma)^n y$

$\Rightarrow y\rho x$. ρ is symmetric.

Suppose, $x\rho y, y\rho z \Rightarrow x\Gamma(y\Gamma)^{n-1}y= (y\Gamma)^n y, y\Gamma(x\Gamma)^{n-1}x= (x\Gamma)^n x \longrightarrow (1)$

also $y\Gamma(z\Gamma)^{n-1}z= (z\Gamma)^n z, z\Gamma(y\Gamma)^{n-1}y= (y\Gamma)^n y \longrightarrow (2)$.

From (2), $(y\Gamma)^{m-1}y\Gamma(z\Gamma)^{n-1}z= (z\Gamma)^{(m+n)-1}z$ for any odd positive integer 'm'.

$\Rightarrow x\Gamma(y\Gamma)^{n-1}y\Gamma(z\Gamma)^{m-1}z=(y\Gamma)^n y\ \Gamma(z\Gamma)^{m-1}z= (z\Gamma)^{m+n}z \Rightarrow x\Gamma((z\Gamma)^{(m+n)-1}z) = (z\Gamma)^{m+n}z$.

Similarly $z\Gamma((x\Gamma)^{(m+n)-1}x) = (x\Gamma)^{m+n}x$ from (1) $\Rightarrow x\rho z$. ρ is transitive.

$\therefore \rho$ is "equivalence relation".

Now let $x\rho y$, $z \in$ T. Since ρ is reflexive $\Rightarrow z\rho z$.

$\therefore x\rho y$ and $z\rho z \Rightarrow (x + z)\rho(y + z)$ and $x\rho y \Rightarrow x\Gamma(y\Gamma)^{n-1}y= (y\Gamma)^n y, y\Gamma(x\Gamma)^{n-1}x= (x\Gamma)^n x$.

To prove that $(x\Gamma u\Gamma z)\ \rho\ (y\Gamma u\Gamma z)$, $(u\Gamma x\Gamma z)\ \rho\ (u\Gamma y\Gamma z)$ & $(u\Gamma z\Gamma x)\ \rho\ (u\Gamma z\Gamma y)$.

Consider $(x\Gamma u\Gamma z)\Gamma(y\Gamma u\Gamma z\Gamma)^{n-1}(y\Gamma u\Gamma z)= (x\Gamma u\Gamma z)\Gamma(y\Gamma)^{n-1}y\Gamma(u\Gamma)^{n-1}u\Gamma(z\Gamma)^{n-1}z$

$=(u\Gamma z)\Gamma(x\Gamma(y\Gamma)^{n-1}y)\Gamma(u\Gamma)^{n-1}u\Gamma(z\Gamma)^{n-1}z$

$= (u\Gamma z)\Gamma(y\Gamma)^n y)\Gamma(u\Gamma)^{n-1}u\Gamma(z\Gamma)^{n-1}z$

$= (y\Gamma)^n y)\Gamma(u\Gamma(u\Gamma)^{n-1}u)\Gamma(z\Gamma(z\Gamma)^{n-1}z\ )$

$= (y\Gamma)^n y)\Gamma(u\Gamma)^n u)\Gamma(z\Gamma)^n z\ )= (y\Gamma u\Gamma z\Gamma)^n(y\Gamma u\Gamma z)$.

$\Rightarrow (x\Gamma u\Gamma z)\ \rho\ (y\Gamma u\Gamma z)$.

Similarly,we can show that $(u\Gamma x\Gamma z)\,\rho\,(u\Gamma y\Gamma z)$& $(u\Gamma z\Gamma x)\,\rho\,(u\Gamma z\Gamma y)$

=> ρ is "congruenceonT".

Let, $(x\Gamma)^2x\rho(x\Gamma x\Gamma y)\rho(y\Gamma y\Gamma x)\rho(y\Gamma)^2y$foranyx,yin$T$.

It follows that$(x\Gamma x\Gamma y)\rho(y\Gamma)^2y$=>$(x\Gamma x\Gamma y)((y\Gamma)^2y\Gamma)^{n-1}(y\Gamma)^2y= (y\Gamma)^2y\Gamma)^n(y\Gamma)^2y$. for even $n\in$ N. Since ρ is congruence wehave$(y\Gamma y\Gamma x)\rho(y\Gamma)^2y$

=>$(y\Gamma y\Gamma x\Gamma x\Gamma x)\rho((y\Gamma)^3x\Gamma x)\rho(y\Gamma y\Gamma x\Gamma y\Gamma y)\rho((x\Gamma x\Gamma y)\Gamma y\Gamma y)\rho((x\Gamma)^2x\Gamma y\Gamma y)\rho(x\Gamma x\,\Gamma(x\Gamma x\Gamma y))\rho(x\Gamma x\Gamma(x\Gamma)^2x\rho(x\Gamma)^4x$.

Therefore,$(y\Gamma y\Gamma x\Gamma x\Gamma x)\rho(x\Gamma)^4x$=>$(y\Gamma y\Gamma x\Gamma x\Gamma x)\Gamma(x\Gamma)^4x)^{n-1}(x\Gamma)^4x$

$=(x\Gamma)^4x\Gamma)^n(x\Gamma)^4x$for even $n\in$N

$\Longrightarrow(y\Gamma y\Gamma(x\Gamma)^4x\Gamma)^n x\Gamma x\Gamma x= (x\Gamma)^4x\Gamma)^n(x\Gamma)^4x$.

$\therefore y\Gamma(x\Gamma)^{n-1}x= (x\Gamma)^nx$.

Similarly we prove that $x\Gamma(y\Gamma)^{n-1}y= (y\Gamma)^ny$ =>$x\rho y$.

ρ is "separative congruence on T".

Definition 3.3: A "ternary Γ-semiring" (T, Γ, +, [ ]) is known as"weakly commutative" if for anyx,y, zin 'T' we have $(x\Gamma y\Gamma z)^{k-1}(x\Gamma y\Gamma z)= x\Gamma a\Gamma b =a\Gamma x\Gamma b = a\Gamma b\Gamma x = a\Gamma y\Gamma b =y\Gamma a\Gamma b =a\Gamma b\Gamma y = a\Gamma b\Gamma z = a\Gamma z\Gamma b = z\Gamma a\Gamma b$for some a,bin 'T'and a odd positive integer k.

Theorem3.4:Let (T, Γ, +, [ ]) bea"weaklycommutativepermutableternary Γ-semiring".Define a relation ρas in the above theorem then ρ is "separative congruence on T".

***Proof*:**Giventhat$T$is "weaklycommutative".Thenforanyx,y,zand for some a, b inTand$k$ isa"odd positiveinteger,"$(x\Gamma y\Gamma z)^{k-1}(x\Gamma y\Gamma z)= x\Gamma a\Gamma b =a\Gamma b\Gamma x = b\Gamma x\Gamma a = a\Gamma x\Gamma b = b\Gamma a\Gamma x = x\Gamma b\Gamma a =a\Gamma y\Gamma b =y\Gamma b\Gamma a =b\Gamma a\Gamma y=y\Gamma a\Gamma b = b\Gamma y\Gamma a =a\Gamma b\Gamma y = a\Gamma b\Gamma z = b\Gamma z\Gamma a = z\Gamma a\Gamma b = b\Gamma a\Gamma z = z\Gamma b\Gamma a = a\Gamma z\Gamma b$.Toprovethat ρ is "separativecongruenceonT". Since by theorem 3.2, ρ is equivalence relation.

Now we prove that ρ is "congruence", let$x\rho y$, $z\in$ T. Since ρ is reflexive=>$z\rho z$.

$\therefore x\rho y$ and $z\rho z\Longrightarrow (x + z)\rho(y + z)$ and$x\,\rho\,y$=>$x\Gamma(y\Gamma)^{n-1}y= (y\Gamma)^ny$,$y\Gamma(x\Gamma)^{n-1}x= (x\Gamma)^nx$.

To prove that $(x\Gamma a\Gamma b)\,\rho\,(y\Gamma a\Gamma b)$, $(a\Gamma x\Gamma b)\,\rho\,(a\Gamma y\Gamma b)$& $(a\Gamma b\Gamma x)\,\rho\,(a\Gamma b\Gamma y)$.

Let Consider $(x\Gamma a\Gamma b)\Gamma(y\Gamma a\Gamma b\Gamma)^{n-1}(y\Gamma a\Gamma b)$

$= (y\Gamma a\Gamma b)\Gamma(y\Gamma a\Gamma b\Gamma)^{n-1}(y\Gamma a\Gamma b) = (y\Gamma a\Gamma b\Gamma)^n(y\Gamma a\Gamma b)$.

Similarly, we can show that $(u\Gamma x\Gamma z)\,\rho\,(u\Gamma y\Gamma z)$& $(u\Gamma z\Gamma x)\,\rho\,(u\Gamma z\Gamma y)$=> ρ is "congruenceon T". Let, $(x\Gamma)^2x\rho(x\Gamma x\Gamma y)\rho(y\Gamma y\Gamma x)\rho(y\Gamma)^2y$foranyx, y in$T$.

It follows that$(x\Gamma x\Gamma y)\rho(y\Gamma)^2y$=>$(x\Gamma x\Gamma y)((y\Gamma)^2y\Gamma)^{n-1}(y\Gamma)^2y= (y\Gamma)^2y\Gamma)^n(y\Gamma)^2y$for even $n\in$ N. Since ρ is congruence wehave$(y\Gamma y\Gamma x)\rho(y\Gamma)^2y$

$\Rightarrow (y\Gamma y\Gamma x\Gamma x\Gamma x)\rho((y\Gamma)^3 x\Gamma x)\rho(y\Gamma y\Gamma x\Gamma y\Gamma y)\rho((x\Gamma x\Gamma y)\Gamma y\Gamma y)\rho((x\Gamma)^2 x\Gamma y\Gamma y)\rho(x\Gamma x \Gamma(x\Gamma x\Gamma y))\rho(x\Gamma x\Gamma(x\Gamma)^2 x\rho(x\Gamma)^4 x$.

Therefore, $(y\Gamma y\Gamma x\Gamma x\Gamma x)\rho(x\Gamma)^4 x \Rightarrow (y\Gamma y\Gamma x\Gamma x\Gamma x)\Gamma(x\Gamma)^4 x)^{n-1}(x\Gamma)^4 x =$

$(x\Gamma)^4 x\Gamma)^n(x\Gamma)^4 x$ for even $n \in N \Longrightarrow (y\Gamma y\Gamma(x\Gamma)^4 x\Gamma)^n x\Gamma x\Gamma x = (x\Gamma)^4 x\Gamma)^n(x\Gamma)^4 x$.

$\therefore y\Gamma(x\Gamma)^{n-1}x = (x\Gamma)^n x$.

Similarly we prove that $x\Gamma(y\Gamma)^{n-1}y = (y\Gamma)^n y \Rightarrow x\rho y$.

ρ is "separative congruence on T".

Theorem3.5:Let (T, Γ, +, [ ]) bea "quasi-commutative ternary Γ-semiring" then T/ρ isa "maximal separative homomorphic image ofT".

Proof: Suppose Ψ be an "arbitrary separative congruence "on a "quasi commutative ternary Γ-semiring".

we first prove that, let x,y in Tif $x\Gamma(y\Gamma)^{n-1}y\,\Psi\,(y\Gamma)^n y$, $y\Gamma(x\Gamma)^{n-1}x\,\Psi\,(x\Gamma)^n x$for a "even positive integer" n then x Ψ y, for $n = 1$, $x\Gamma y\,\Psi\, y\Gamma y$, $y\Gamma x\,\Psi\, x\Gamma x$then x Ψ y.

Assume that the results holds for $n \geq 1$, $x\Gamma(y\Gamma)^n y\,\Psi\,(y\Gamma)^{n+1}y$, $y\Gamma(x\Gamma)^n x\,\Psi\,(x\Gamma)^{n+1}x$.

Now $[x\Gamma(y\Gamma)^{n-1}y]\Gamma[x\Gamma(y\Gamma)^{n-1}y]\Psi((y\Gamma)^n y)\Gamma((y\Gamma)^n y) \Longrightarrow x\Gamma(y\Gamma)^{n-1}y\Psi(y\Gamma)^n y$.

Similarly, weprove$y\Gamma(x\Gamma)^{n-1}x\Psi(x\Gamma)^n x$. By induction x Ψ y.

Since Λ is "congruence relation" on T.

We have $x\Lambda y \Rightarrow x\Gamma(y\Gamma)^{n-1}y = (y\Gamma)^n y$, $y\Gamma(x\Gamma)^{n-1}x = (x\Gamma)^n x$for a "even positive integer n"

$\Rightarrow x\Gamma(y\Gamma)^n y\,\Psi\,(y\Gamma)^{n+1}y$, $y\Gamma(x\Gamma)^n x\,\Psi\,(x\Gamma)^{n+1}x \Rightarrow x\,\Psi\, y \Rightarrow \Lambda \subseteq \psi$. By "homomorphic" theorem it is easy to see that T/ρ is a "maximal homomorphic image of$T$".

Theorem 3.6: If (T, Γ, +, [ ]) be a "ternary Γ-semiring" and let 'Ψ' be any "congruence relationon T" then Ψ is a "maximal separative congruence T".

Proof: Let Ψ be an "arbitrary separative congruence" on a "ternary Γ-semiring".
we first prove the following Lemma. Let x,yin Tif $x\Gamma(y\Gamma)^{n-1}y\,\Psi\,(y\Gamma)^n y$, $y\Gamma(x\Gamma)^{n-1}x\,\Psi\,(x\Gamma)^n x$for a even positive integer$n$then$x\,\Psi\, y$ for$n$=2,$x\Gamma y\Gamma y\,\Psi\,(y\Gamma)^2 y$,$y\Gamma x\Gamma x\,\Psi\,(x\Gamma)^2 x$thenx Ψ y.
Assumethattheresultsholds for odd $n \geq 1$,$x\Gamma(y\Gamma)^n y\,\Psi(y\Gamma)^{n+1}y$,$y\Gamma(x\Gamma)^n x\,\Psi(x\Gamma)^{n+1}x$.
Now $(x\Gamma(y\Gamma)^{n-1}y)\Gamma(x\Gamma(y\Gamma)^{n-1}y)\,\Psi\,(y\Gamma)^{2n+1}y \Rightarrow x\Gamma(y\Gamma)^{n-1}y\,\Psi\,(y\Gamma)^n y$.

Similarly we prove $y\Gamma(x\Gamma)^{n-1}x\,\Psi\,(x\Gamma)^n x$. By induction x Ψ y.

Let ρ be an "arbitrary separative congruence" on a "ternary Γ-semiring".

If $x\rho y \Rightarrow x\Gamma(y\Gamma)^{n-1}y\,\Psi\,(y\Gamma)^n y$, $y\Gamma(x\Gamma)^{n-1}x\,\Psi\,(x\Gamma)^n x$for a even positive integer n and so $x\Gamma(y\Gamma)^{n-1}y$ $\Psi\,(y\Gamma)^n y$, $y\Gamma(x\Gamma)^{n-1}x\,\Psi\,(x\Gamma)^n x$then x Ψ y$\Rightarrow \rho \subseteq \Psi$.

Hence Ψ is a "maximal separative congruence" on$T$.

Definition 3.7: A "ternary Γ-semiring" (T, Γ, +, [ ]) is known as "***ρ-reflexive***" if $\forall x, y, z \in T$ ∃ an odd $m \in N$ ∋ $(x\Gamma y\Gamma z) = (y\Gamma z\Gamma x\Gamma)^{m-1}(y\Gamma z\Gamma x)$ or $(x\Gamma y\Gamma z) = (z\Gamma x\Gamma y\Gamma)^{m-1}(z\Gamma x\Gamma y)$.

Definition 3.8: A "ternary Γ-semiring" (T, Γ, +, [ ]) is said to be "***permutable***" if for every x, y, z in T, $x\Gamma y\Gamma z = x\Gamma z\Gamma y = y\Gamma x\Gamma z$.

Theorem 3.9: Let (T, Γ, +, [ ]) is "ρ-reflexive", "separative" and "permutable ternary Γ-semiring" then (T, Γ, +, [ ]) is "cancellative".

Proof: Given that (T, Γ, +, [ ]) is ρ-reflexive,

for every x, a, bin T, $x\Gamma a\Gamma b= (a\Gamma b\Gamma x\Gamma)^{m-1}(a\Gamma b\Gamma x)$ and $y\Gamma a\Gamma b=(a\Gamma b\Gamma y\Gamma)^{n-1}(a\Gamma b\Gamma y)$ forsome odd positiveinteger$m$,n.Also$T$is "permutable","separative".

To prove that T is "cancellative" that is $x\Gamma a\Gamma b= y\Gamma a\Gamma b \Rightarrow x = y$.

Let $x\Gamma a\Gamma b = y\Gamma a\Gamma b$.

Consider,$x\Gamma a\Gamma b= (a\Gamma b\Gamma x\Gamma)^{m-1}(a\Gamma b\Gamma x)= (a\Gamma)^{m-1}a\Gamma(b\Gamma)^{m-1}b\Gamma(x\Gamma)^{m-1}x$

$\Rightarrow (x\Gamma)^{m-1}x = x\Gamma(a\Gamma)^{1-m}(b\Gamma)^{-m}b$

and $y\Gamma a\Gamma b =(a\Gamma b\Gamma y\Gamma)^{n-1}(a\Gamma b\Gamma y)= a\Gamma)^{m-1}a\Gamma(b\Gamma)^{m-1}b\Gamma(y\Gamma)^{m-1}y$

$\Rightarrow(y\Gamma)^{m-1}y= y\Gamma(a\Gamma)^{1-m}(b\Gamma)^{-m}b$

Consider $(x\Gamma)^{m}x = (x\Gamma)^{m-1}x\Gamma x = x\Gamma(a\Gamma)^{1-m}(b\Gamma)^{-m}b\Gamma x$

$= x\Gamma(a\Gamma)^{1-m}(b\Gamma)^{1-m}x= (x\Gamma a)\Gamma(a\Gamma)^{-m}\Gamma(b\Gamma)^{1-m}x$

$= (x\Gamma a\Gamma b)\Gamma(a\Gamma)^{-m}\Gamma(b\Gamma)^{-m}\Gamma x= (y\Gamma a\Gamma b)\Gamma(a\Gamma)^{-m}\Gamma(b\Gamma)^{-m}\Gamma x$

$= (y\Gamma a)\Gamma(a\Gamma)^{-m}\Gamma(b\Gamma)^{1-m}x= y\Gamma(a\Gamma)^{1-m}(b\Gamma)^{1-m}x$

$= y\Gamma(a\Gamma)^{1-m}(b\Gamma)^{-m}b\Gamma x= (y\Gamma)^{m-1}y\Gamma x = (y\Gamma)^{m}x$

Now for odd $m > 1$ $(x\Gamma)^{2m-2}x\Gamma y = (x\Gamma)^{m-2}(x\Gamma)^{m+1}y = (x\Gamma)^{m-2}(x\Gamma)^{m}y\Gamma y = (x\Gamma)^{2m-2}y\Gamma y$

$= (x\Gamma)^{m-1}y\Gamma(x\Gamma)^{m-1}y\Gamma(x\Gamma)^{2m-2}x\Gamma y = (x\Gamma)^{m-1}(x\Gamma)^{m}\Gamma y$

$= (x\Gamma)^{m-1}(x\Gamma)^{m}x = (x\Gamma)^{m-1}x\Gamma(x\Gamma)^{m-1}x.$

$\Longrightarrow(x\Gamma)^{m-1}y\Gamma(x\Gamma)^{m-1}y = (x\Gamma)^{m-1}x\Gamma(x\Gamma)^{m-1}x$

$\Longrightarrow(x\Gamma)^{m-1}y = (x\Gamma)^{m-1}x.$

For $m = 3$, $(x\Gamma)^2x= x\Gamma x\Gamma y$. Similarly we prove that $(y\Gamma)^2y= y\Gamma y\Gamma x$
$\Rightarrow(x\Gamma)^2x= x\Gamma x\Gamma y= y\Gamma y\Gamma x = (y\Gamma)^2y \Rightarrow x = y$. Therefore ($T$, Γ, +, [ ])is cancellative.

Theorem 3.10:Let (T, Γ, +, [ ]) is ρ-reflexive and permutable "ternary Γ-semiring" then (T, Γ, +, [ ]) is cancellative.

Proof: Similar to theorem 3.9.

4. "Idempotent Pair" in "Ternary Γ-Semirings"

Definition 4.1: A pair (u, v) of "elements" in a "ternary Γ-semiring" T is known as an "***idempotent pair***" if $u\Gamma v\Gamma(u\Gamma v\Gamma x) = u\Gamma v\Gamma x$ and $(x\Gamma u\Gamma v)\Gamma u\Gamma v = x\Gamma u\Gamma v \forall x \in T$.

Definition4.2:A pair (u, v) of "elements" in a "ternary Γ-semiring" T is known as a "***natural idempotent pair***" if $u\Gamma v\Gamma u = u$ and $v\Gamma u\Gamma v = v$.

Definition4.3: "Two idempotent pairs" (u, v) and (w, z) of T are said to be "***commute***"if $u\Gamma v\Gamma(w\Gamma z\Gamma x) = w\Gamma z\Gamma(u\Gamma v\Gamma x)$ and $(x\Gamma u\Gamma v)\Gamma w\Gamma z= (x\Gamma w\Gamma z)\Gamma u\Gamma v\ \forall x \in T$.

Definition4.4:"Two idempotent pairs"(u, v) and (w, z) of T are said to be "***equivalent***" if $u\Gamma v\Gamma x = w\Gamma z\Gamma x$ and $x\Gamma u\Gamma v = x\Gamma w\Gamma z\ \forall x \in T$. In notion we write as $(u,v) \sim (w, z)$.

Clearly, the relation defined above is an equivalence relation. The equivalence class containing the idempotent pair (u, v) is denoted by $\overline{(u,v)}$. E(T) denote the set of all "equivalence classes" of"idempotent pairs" in T.

Theorem 4.5: Every "idempotent pair" (u,v) in T is "equivalent" to a "natural idempotent pair" (w, z) in T.

Proof : Suppose $w = u\Gamma v\Gamma u$ and $z = v\Gamma u\Gamma v$.
Then $w\Gamma z\Gamma w = (u\Gamma v\Gamma u)\Gamma(v\Gamma u\Gamma v)\Gamma(u\Gamma v\Gamma u) = u\Gamma v\Gamma u = w$ and
$z\Gamma w\Gamma z = (v\Gamma u\Gamma v)\Gamma(u\Gamma v\Gamma u)\Gamma(v\Gamma u\Gamma v) = v\Gamma u\Gamma v = z$, since (u,v) is an "idempotent pair" in T.
Again,$w\Gamma z\Gamma x = (u\Gamma v\Gamma u)\Gamma(v\Gamma u\Gamma v)\Gamma x = u\Gamma v\Gamma(u\Gamma v\Gamma u)\Gamma v\Gamma x = u\Gamma v\Gamma(u\Gamma v\Gamma x)= u\Gamma v\Gamma x$ and
$x\Gamma w\Gamma z = x\Gamma(u\Gamma v\Gamma u)\Gamma(v\Gamma u\Gamma v) =x\Gamma u\Gamma(v\Gamma u\Gamma v)\Gamma u\Gamma v = (x\Gamma u\Gamma v)\Gamma u\Gamma v = x\Gamma u\Gamma v$.

Now assume that T is a "ternary Γ-semiring" in which all the "idempotent pairs commute" mutually. If (u, v) and (w, z) are "two idempotent pairs" in T, then $(u\Gamma v\Gamma w, z)$ is also an "idempotent pair" in T, because

$$\begin{aligned}(u\Gamma v\Gamma w\Gamma z)\Gamma(u\Gamma v\Gamma w\Gamma z\Gamma x) &= (u\Gamma v\Gamma w\Gamma z)\Gamma(w\Gamma z\Gamma u\Gamma v\Gamma x)\\ &= u\Gamma v\Gamma(w\Gamma z\Gamma u\Gamma v\Gamma x)\\ &= u\Gamma v\Gamma\ (u\Gamma v\Gamma w\Gamma z\Gamma x) = u\Gamma v\Gamma w\Gamma z\Gamma x\ \forall x\in T \text{ and}\end{aligned}$$

$$\begin{aligned}(x\Gamma u\Gamma v\Gamma w\Gamma z)\Gamma(u\Gamma v\Gamma w\Gamma z) &= (x\Gamma w\Gamma z\Gamma u\Gamma v)(u\Gamma v\Gamma w\Gamma z)\\ &= (x\Gamma w\Gamma z\Gamma u\Gamma v)\Gamma w\Gamma z\\ &= (x\Gamma u\Gamma v\Gamma w\Gamma z)\Gamma w\Gamma z = x\Gamma u\Gamma v\Gamma w\Gamma z\ \forall x\in T.\end{aligned}$$

Similarly, we can show that $(u, v\Gamma w\Gamma z)$ is also an "idempotent pair" in T.
Moreover, $(u\Gamma v\Gamma w, z) \sim (u, v\Gamma w\Gamma z)$.

Definition4.6: A "commutative idempotent Γ-semigroup" is called a "***semi-lattice***".

Theorem 4.7: Let T be a "ternary Γ-semiring" in which all the "idempotent pairs" "commute mutually". Then E(T) is a "semi-lattice" under the (binary) product defined by $(u, v).(w, z) = (u\Gamma v\Gamma w, z)$.

Proof:In E(T)**,** we show that the above definition of product is well-defined. Suppose that $\overline{(u,v)} = \overline{(u',v')}$ and $\overline{(w,z)} = \overline{(w',z')}$ in E(S). Then $u\Gamma v\Gamma x = u'\Gamma v'\Gamma x$, $x\Gamma u\Gamma v = x\Gamma u'\Gamma v'$ and

$w\Gamma z\Gamma x= w'\Gamma z'\Gamma x$, $x\Gamma w\Gamma z= x\Gamma w'\Gamma z'$ for all $x\ni$T. Hence $(u\Gamma v\Gamma w)\Gamma z x = (u'\Gamma v'\Gamma w)\Gamma z\Gamma x = u'\Gamma v'\Gamma(w\Gamma z\Gamma x) = u'\Gamma v'\Gamma(w'\Gamma z'\Gamma x) \Rightarrow (u\Gamma v\Gamma w)\Gamma z\Gamma x= (u'\Gamma v'\Gamma w')\Gamma z'\Gamma x$ and similarly, $x\Gamma(u\Gamma v\Gamma w)\Gamma z= x\Gamma(u'\Gamma v'\Gamma w')\Gamma z'$ $\forall x\in$ T, hence, $(u\Gamma v\Gamma w,z) \sim (u'\Gamma v'\Gamma w', z')$ i.e. $(u\Gamma v\Gamma w,z) = (u'\Gamma v'\Gamma w', z')$. Thus the product in E(T) is "well-defined". Clearly, the "associative property holds" in E(T). Hence E(T) is a "Γ-semigroup". Since by "hypothesis", the "idempotent pairs" in T"commute", E(T) is "commutative". Further, $\overline{(u,v)}.\overline{(u,v)} = \overline{(u\Gamma v\Gamma u,v)} = \overline{(u,v)}$, since $\overline{(u\Gamma v\Gamma u,v)} : \overline{(u,v)}$. This shows that every "element" of E(T) is "idempotent". Thus E(T) is a "semi-lattice".

Remark 4.8:The "ssociated partial ordering" in the "semi-lattice" E(T) is given by $(u,v) <(w,z)$ if $(u, v).(w,z) = (u,z)$ and is called the "natural partial ordering" of E(T).

Theorem 4.9: Let T be a "multiplicatively cancellative (MC) ternary Γ-semiring". Then the "idempotent pairs" of "elements" of T (if they exist) are all "equivalent".

Proof: Let (u, v) be an "idempotent pair" in a "MC ternary Γ-semiring T". Then$u\Gamma v\Gamma(u\Gamma v\Gamma x) = u\Gamma v\Gamma x$that $u\Gamma v\Gamma x= x \forall x\in$T, by "MLC".Hence, for any "two idempotent pairs" (u, v) and (w, z), we have $u\Gamma v\Gamma x= x= w\Gamma z\Gamma x \forall x \in$ T. Similarly, $x\Gamma u\Gamma v= x = x\Gamma w\Gamma z \forall x \in$ T. Consequently, $(u, v) \sim (w, z)$.

References

[1] Bedrich Pondelicek, (1973), "On Weakly Commutative Semigroups" .

[2] Chacron M., and Thierrin G., (1972), "σ-Reflexive Semigroup and Rings",Can. Math. Bull., 15, 185-88.

[3] N. P. Mukherjee, (1972), "Quasi Commutative Semigroups I", Czech Slovak Math. Journal., 22, 449-453.

[4] E. Hewitt, and H. S. Zuckerman, "The L1-Algebra of Commutative Semigroup"

[5] M. Vasantha, Theory of T**Γ**-Ideals &**Γ**-TS-Act over Ternary **Γ**-Semigroups – Ph.D. Thesis, 2019, K.L. University, Vaddeswaram, A.P. India.

[6] M. Vasantha, D. MadhusudhanaRao and M. VenkateswaraRao- Structure and Study of Elements in Ternary **Γ**-Semigroups-International Journal of Engineering Research,Volume 4, Issue 4, 1st April-2015, pp 197-202.

[7] MamillapalliVasantha, D. MadhusudhanaRao, P. Siva Prasad, B. Srinivasa Kumar, T. Satish – On Γ-TS acts Over ternary Γ-semigroups, International journal of Engineering & Technology, 7(4.10) 2018, 812-815.

[8] M. SajaniLavanya, Classification And Study Of Ternary **Γ**-Ideals In Ternary **Γ**-Semirings Ph.D. Thesis, July 2017, AcharyaNagarjuna University, Nagarjuna University, Nagarnuna Nagar, Guntur.

[9] M. SajaniLavanya, Dr. D. MadhusudhanaRaoand V. Syam Julius Rajendra, On Lateral Ternary Γ-Ideals of Ternary Γ-Semirings, American International Journal of Research in Science, Technology, Engineering & Mathematics, 12(1), September-November, 2015, pp. 11-14.

* * * * *