RELATIONS BETWEEN TOPOLOGY AND COVERING SPACE

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ABSTRACT

It has been seen that any convex subspace of $\mathbb{R}^n$ has a trivial fundamental group; the authors have computed some fundamental groups that are not trivial. One of the most useful tools for this purpose is the notion of *covering space*, which have been introduced in this paper. Covering spaces are also important in the study of Riemann surfaces and complex manifolds. The authors have shown the relationship between topology and covering spaces by giving many proofs and examples.

Keywords: topology, covering spaces, homomorphism, covering transformation, existence, etc.

1. THE UNIVERSAL COVERING SPACE

Suppose $p : E \to B$ is a covering map, with $p(e_0) = b_0$. If E is simply connected, then E is called a ***universal covering space*** of B. Since $\pi_1(E, e_0)$ is trivial, this covering space corresponds to the trivial subgroup of $\pi_1(B, b_0)$ under the correspondence defined in the preceding section. Theorem thus implies that any two universal covering spaces of B are equivalent. For this reason, we often speak of "the" universal covering space of a given space B. Not every space has a universal covering space, as we shall see. For the moment, we shall simply assume that B has a universal covering space and derive some consequences of this assumption [1-3].

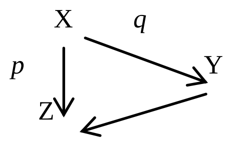
Lemma 1.1 Let B be path connected and locally path connected. Let $p : E \to B$ be a covering map in the former sense (so that E is not required to be path connected). If E_0 is a path component of E, then the map $p_o : E_0 \to B$ obtained by restricting p is a covering map.

Proof. We first show p_0 is surjective. Since the space E is locally homeomorphic to B, it is locally path connected. Therefore E_0 is open in E. It follows that $p(E_0)$ is open in B. We show that $p(E_0)$ is also closed in B, so that $p(E_0) = B$.

Let x be a point of B belonging to the closure to the closure of $p(E_0)$. Let U be a path-connected neighborhood of x that is evenly covered by p. Since U contains a point of $p(E_0)$, some slice V_α of $p^{-1}(U)$ must intersect E_0. Since V_α is homeomorphic to U, it is path connected; therefore it must be contained in E_0. Then $p(V_\alpha) = U$ is contained in $p(E_0)$, so that in particular, $x \in p(E_0)$.

Now we show $po : E_0 \to B$ is a covering map. Given $x \in B$, choose a neighborhood U of x as before. If V_α is a slice of $p^{-1}(U)$, then V_α is path connected; if it intersects E_0, it lies in E_0. Therefore $p_0^{-1}(U)$ equals the union of those slices V_α of $p^{-1}(U)$ that intersect E_0; each of these is open in E_0 and is mapped homeomorphically by p_0 onto U. Thus U is evenly covered by p_0.

Lemma 1.2 Let *p*, *q*, and r be continuous maps with $p = r \circ q$, as in the following diagram:



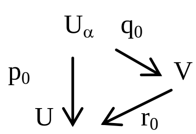
(a) If *p* and r are covering maps, so is *q*.

* (b) If *p* and r are covering maps, so is *r*.

Proof. By our convention X, Y, and Z are path connected and locally path connected. Let $x_0 \in X$; set $y_0 = q(x_0)$ and $z_0 = p(x_0)$

(a) Assume that *p* and *r* are covering maps. We show first that *q* is surjective Given $y \in Y$, choose a path $\tilde{\alpha}$ in Y from y_0 to y. Then $\alpha = r \circ \tilde{\alpha}$ is a path in Z beginning at z_0; let $\tilde{\tilde{\alpha}}$ be a lifting of α to a path in X beginning at x_0. Then $q \circ \tilde{\tilde{\alpha}}$ is a lifting of α to Y that begins at y_0. By uniqueness of path liftings, $\tilde{\alpha} = q \circ \tilde{\tilde{\alpha}}$. Then *q* maps the end point of $\tilde{\tilde{\alpha}}$ to the end point y of $\tilde{\alpha}$. Thus *q* is surjective.

Given $y \in Y$, we find a neighborhood of y that is evenly covered by *q*, Let z = r(y). Since *p* and *r* are covering maps, we can find a path-connected neighborhood U of z that is evenly covered by both *p* and r. Let V be the slice of $r^{-1}(U)$ that contains the point y; we show V is evenly covered by *q*. Let $\{U\alpha\}$ be the collection of slices of $p^{-1}(U)$. Now *q* maps each set U_α into the set $r^{-1}(U)$; because U_α is connected it must be mapped by *q* into a single one of the slices of $r^{-1}(U)$. Therefore, q^{-1} (V) equals the union of those slices U_α that are mapped by *q* into V. It is easy to see that each such U_α is mapped homeomorphically onto V by *q*. For let p_0, q_0 r_0 be the maps obtained by restricting *p*, *q*, and r, respectively, as indicated in the following diagram:



Because p_0 and r_0 are homeomorphisms, so is $q_0 = r_0^{-1} \circ p_0$.

∗(b) We shall use this result only in the exercises. Assume that *p* and *q* are covering maps. Because $p = r \circ q$ and *p* is surjective, r is also surjective.

Given $z \in Z$, let U be a path-connected neighborhood of z that is evenly covered by *p*. We show that U is also evenly covered by r. Let $\{V_\beta\}$ be the collection of path components of r^{-1} (U); these sets are disjoint and open in Y. We show that for each β the map *r* carries V_β homeomorphically onto U.

Let $\{U_\alpha\}$ be the collection of slices $p^{-1}(U)$; they are disjoint, open, and path connected, so they are the path components of p^{-1} (U). Now *q* maps each U_α into the set r^{-1} (U); because U_α is connected, it must be mapped by *q* onto one of the sets V_β. Therefore q^{-1} (V_β) equals the union of a subcollection of the collection $\{U_\alpha\}$. Theorem and

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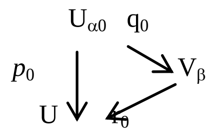
Lemma together imply that if $U_{\alpha 0}$ is any one of the path components of q^{-1} (V_β) then the map $q_0 : U_{\alpha 0} \to V_\beta$ obtained by restricting q is a covering map.

In particular, q_0 is surjective. Hence q_0 is a homeomorphism, being continuous, open and injective as well. Consider the maps.



obtained by restricting p, q, and r. Because p_0 and q_0 are homeomorphism, so is r_0.

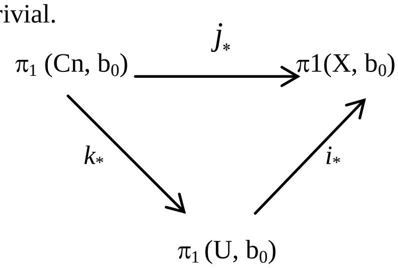
Lemma 1.3 Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. If E is simply connected, then b_0 has a neighborhood U such that inclusion $i : U \to B$ induces the trivial homomorphism

$$i_* : \pi_1 (U, b_0) \to \pi_1 (B, b_0)$$

Proof. Let U be a neighborhood of b_0 that is evenly covered by p; break p^{-1} (U) up into slices; let U_α be the slice containing e_0. Let f be a loop in U based at b_0. Because p defines a homeomorphism of U_α with U, the loop f lifts to a loop $\tilde{f}$ in U_α based at e_0. Since E is simply connected, there is a path homotopy $\tilde{F}$ in E between $\tilde{f}$ and a constant loop. Then $p \circ \tilde{F}$ is a path homotopy in B between f and a constant loop.

Example 1. Let X be our familiar "infinite earning" in the plane; if C_n is the circle of radius 1/n in the plane with centre at the point (1/n, 0), then X is the union of the circles C_n. Let b_0 be the origin; we show that if U is any neighborhood of b_0 in X, then hte homomorphism of fundamental groups induced by inclusion $i : U \to X$ is not trivial.

Given n, there is a retraction $r : X \to C_n$ obtained by letting r map each circle C_i for $i \neq n$ to the point b_0. Choose n large enough that C_n lies in U. Then in the following diagram of homomorphisms induced by inclusion, j_* is injective; hence i_* cannot be trivial.



It follows that even through X is path connected and locally path connected, it has no universal covering space.

2. COVERING TRANSFORMATIONS

Given a covering map $p : E \to B$, it is of some interest to consider the set of all equivalences of this covering space with itself. Such an equivalence is called a ***covering transformation***. Composites and inverses of covering transformations are covering transformations, so this set forms a group; it is called the ***group of covering transformations*** and denoted *C* (E, p, B) [4-6].

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Throughout this section, we shall assume that $p : E \to B$ is a covering map with $p(e_0) = b_0$; and we shall let $H_0 = p_*(\pi_1 (E, e_0))$. We shall show that the group C (E, p, B) is completely determined by the group π_1 (B, b_0) and the subgroup H_0. Specifically, we shall show that if $N(H_0)$ is the largest subgroup of π_1 (B, b_0) of which H_0 is a normal subgroup. C (E, p, B) is isomorphic to N (H_0) / H_0.

Definition 2.1. If H is a subgroup of the group G, then the ***normalizer*** of H in G is the subset of G defined by the equation. $N(H) = [g]\ gHg^{-1} = H\}$

It is easy to see that N(H)is a subgroup of G. It follows from the definition that it contains H as a normal subgroup and is the largest such subgroup of G.

Definition 2.2. Given p : E $\to$ B with $p(e_0) = b_0$, let F be the set F = p–1 (e_0).

Let $\Phi : \pi_1 (B, b_0) / H_0 \to F$

be the lifting correspondence of Theorem 3.21; it is a bijection. Define also a correspondence.

$\psi : C(E, p, B) \to F$

by setting $\psi(h) = (e_0)$ for each covering transformation $h : E \to E$. Since h is uniquely determined once its value at e_0 is known, the correspondence ψ is injective.

Lemma 2.3. The image of the map ψ equals the image under Φ of the subgroup $N(H_0) / H_0$ of $\pi_1 (B, b_0) / H_0$.

Proof. Recall that the lifting correspondence $\phi : \pi_1 (B, b_0) \to F$ is defined as follows: Given a loop α in B at b_0, let γ be its lift to E beginning at e_0; let $e_1 = \gamma(1)$; and define ϕ by setting $\phi([\alpha]) = e_1$. To prove the lemma, we need to show that there is a covering transformation $h : E \to E$ with $h(e_0) = e_1$ if and only if $[\alpha] \in N(H_0)$.

This is easy. Lemma 3.41 tells us that h exists if and only if $H_0 = H_1$, where $H_1 = p_* (\pi_1 E, e_1)$. And Lemma 3.43 tells us that $[\alpha] * H_1 * [\alpha]^{-1} = H_0$. Hence h exists if and only if $[\alpha] * H_0 * [\alpha]^{-1} = H_0$, which is simply the statement that $[\alpha] \in N(H_0)$.

Theorem 2.4 The bijection $\Phi^{-1} \circ \psi : C(E, p, B) \to N(H_0) / H_0$ is an isomorphism of groups.

Proof. We need only show that $\Phi^{-1} \circ \psi$ is a homomorphism. Let h, k : E $\to$ E be covering transformations. Let $h(e_0) = e_1$ and $k(e_0) = e_2$; then $\psi(h) = e_1$ and $\psi(k) = e_2$.

by definition. Choose paths γ and δ in E from e_0 to e_1 and e_2, respectively. If $\alpha = p \circ \gamma$ and $\beta = p \circ \delta$, then

$\Phi([\alpha]) H_0) = e_1$ and $\Phi([\beta])H_0) = e_2$,

by definition. Let $e_3 = h(k(e0))$; then $\psi(h \circ k) = e_3$. We show that

$\Phi([\alpha * \beta] H_0) = e_3$. and the proof is complete.

Since δ is a path from e_0 to e_2, the path $h \circ \delta$ is a path from $h(e_0) = e_1$ to $h(e_2) = h(k(e_0)) = e_3$. See Figure 1. Then the product $\gamma * (h \circ \delta)$ is defined and is a path from e_0 to e_3. It is a lifting of $\alpha * \beta$, since $p \circ \gamma = \alpha$ and $p \circ h \circ \delta = p \circ \delta = \beta$. Therefore $\Phi([\alpha * \beta]H_0 = e_3$, as desired.

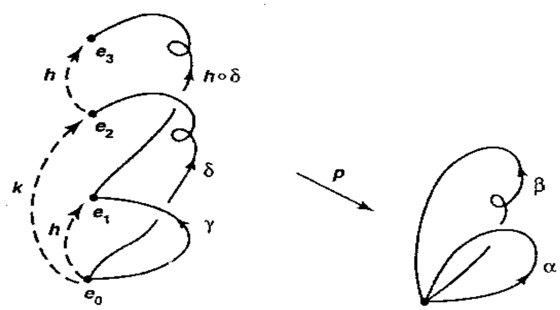

Figure 1

Corollary 2.5. The group H_0 is a normal subgroup of π_1 (B, b_0) if and only if for every pair of points e_1 and e_2 of p^{-1} (b_0), there is a covering transformation h : E → E with $h(e_1) = e_2$. In this case, there is an isomorphism

$$\Phi^{-1} \text{ o } \psi : C\,(E, p, B) \to \pi_1\,(B, b_0) \mid H_0$$

Corollary 2.6. Let p : E → B be a covering map. If is simply connected, then

$$C\,(E, p, B) \cong \pi_1\,(B, b_0)$$

If h_0 is a normal subgroup of π_1 (B, b_0), then p : E → B is called a regular covering map. (Here is another example of the overuse of familiar terms. The words "normal" and "regular" have already been used to mean quite different things!)

Example 2. Because the fundamental group of the circle is abelian, every covering of S^1 is regular. If p : ¡ → S^1 is the standard covering map, for instance, the covering transformation are the homeomorphisms $x \to x + n$. The group of such transformation is isomorphic to Z..

Theorem 2.7. Let X be path connected and locally path connected; let G be a group of homeomorphism of X. The quotient map π : X → X/G is a covering map if and only if the action of G is properly discontinuous. In this case, the covering map π is regular and G is its group of covering transformations.

Proof. We show π is an open map. If U is open in X, then π^{-1} π(U) is the union of the open sets g(U) of X, for $g \in$ G. Hence π^{-1} π(U) is open in X, so that π(U) is open in X/G by definition. Thus π is open.

Step 1. We suppose that the action of G is properly discontinuous and show that π is a covering map. Given $x \in$ X, let U be a neighborhood of x such that g_0(U) and g_1(U) are disjoint whenever $g_0 \neq g_1$. Then π(U) is evenly covered by π. Indeed, π^{-1} π(U) equals the union of the disjoint open sets g(U), for $g \in$ G, each of which contains at most one point of each orbit. Therefore, the map g(U) → π(U) obtained by restricting π is bijective; being continuous and open, it is a homeomorphism. The set g(U), for $g \in$ G, thus form a partition of π^{-1} π(U) into slices.

Step 2. We suppose now that π is a covering map and show that the action of G is properly discontinuous. Given $x \in$ X, let V be a neighborhood of π(x) that is evenly covered by π. Partition π^{-1} (V) into slices; let U_α be the slice containing x. Given $g \in$ G with $g \neq$ e, the set $g(U_\alpha)$ must be disjoint from U_α, for otherwise, two points of U_α would belong to the same orbit and the restriction of π to U_α would not be injective. It follows that the action of G is properly discontinuous.

Step 3. We show that if π is a covering map, then G is its group of covering transformations and π is regular. Certainly any $g \in$ G is a covering transformation for π o g = π because the orbit of $g(x)$ equals the orbit of x. On the other hand, let h be a covering transformation with $h(x_1) = x_2$, say. Because π o h = π, the points x_1 and x_2 map to the same point under π; therefore there is an element $g \in$ G such that $g(x_1) = x_2$. The uniqueness part of Theorem then implies that $h = g$.

It follows that π is regular. Indeed, for any two points x_1 and x_2 lying in the same orbit, there is an element $g \in$ G such that $g(x_1) = x_2$. The corollary applies.

Theorem 2.8. If p : X → B is a regular covering map and G is its group of covering transformation, then there is a homeomorphism k : X/G → B such that $p = k$ o π, where π : X → X/G is the projection.

$$\begin{array}{ccc} X & = & X \\ \downarrow \pi & & \downarrow p \\ X/G & \xrightarrow{k} & B \end{array}$$

Proof. If g is a covering transformation, then $p(g(x)) = p(x)$ by definition. Hence p is constant on each orbit, so it induces a continuous k of the quotient space X/G into B. On the other hand, p is a quotient map because it is continuous, surjective, and open. Because p is regualr, any two points of p^{-1} (b) belong to the same orbit under the action of G. Therefore, π induces a continuous map B → X/G that is an inverse for k.

Example 3. Let X be the cylinder $S^1 \times 1$; let h : X → X be the homeomorphism $h(x, t) = (-x, t)$; and let k : X → X be the homeomorphism $k(x, t) = (-x, 1 - t)$. The groups $G_1 = \{e, h\}$ and $G_2 = \{e, k\}$ are isomorphic to the integers modulo 2; both act properly discontinuously on X. But X/G_1 is homeomorphic to X, while X/G_2 is homeomorphic to the Mobius band, as you can check. See Figure 2.

Figure 2

3. EXISTENCE OF COVERING SPACES

We have shown that corresponding to each covering map p : E → B is a conjugacy class of subgroups of $\pi_1(B, b_0)$ and that two such covering maps are equivalent if and only if they correspond to the same such class. Thus, we have an injective correspondence from equivalence classes of coverings of B to conjugacy classes of subgroups of $\pi_1(B, b_0)$. Now we ask the question whether this correspondence is surjective, that is, whether for every conjugacy class of subgroups of $\pi_1(B, b_0)$, there exists a covering of B that corresponds to this class [7-9].

Definition 3.1. A space B is said to be ***semilocally simply connected*** if for each $b \in B$, there is a neighborhood U of b such that the homomorphism

$$i* : \pi_1 (U, b) \to \pi_1 (B, b) \qquad \text{induced by inclusion is trivial.}$$

Note that if U satisfies this condition, then so does any smaller neighborhood of b, so that b bas "arbitrarily small" neighborhoods satisfying this condition. Note also that this condition is weaker than true local simple connectedness, which would require that within each neighborhood of b there should exist a neighborhood U of b that is itself simply connected.

Semilocal simple connectedness of B is both necessary and sufficient for there to exist, for every conjugacy class of subgroups of $\pi_1 (B, b_0)$ a corresponding covering space of B. Necessity was proved in Lemma ; sufficiency is proved in this section.

Theorem 3.2. Let B be path connected, locally path connected and semilocally simply connected. Let $b_0 \in B$. Given a subgroup H of $\pi_1 (B, b_0)$ there exists a covering map $p : E \to B$ and points $e_0 \in p^{-1} (b_0)$ such that

$$P_* (\pi_1 (E, e0)) = H.$$

Proof. **Step 1**. Construction of E. The procedure for constructing E is reminiscent of the procedure used in complex analysis for constructing Riemann surfaces. Let .P denote the set of all paths in B beginning at b_0. Define an equivalence relation on *.P* by setting $\alpha \sim \beta$ if α and β end at the same point of B and

$$[\alpha * \bar{\beta}] \in H.$$

This relation is easily seen to be an equivalence relation. We will denote the equivalence class of the path α by $\alpha^\#$. Let E denote the collection of equivalence classes, and define $p: E \to B$ by the equation

$$p(\alpha^\#) = \alpha(1).$$

Since B is path connected, p is surjective. We shall topologize E so that p is a covering map.

We first note two facts :

(a) If $[\alpha] = [\beta]$, then $\alpha^\# = \beta^\#$.

(b) If $\alpha^\#=\beta^\#$, then $(\alpha * \delta)^\# = (\beta * \delta)^\#$ for any path δ in B beginning at $\alpha(1)$.

The first follows by noting that if $[\alpha] = [\beta]$, then $[\alpha * \bar{\beta}]$ is the identity element, which belongs to H. The second follows by noting that $\alpha * \beta$ and $\beta * \delta$ end at the same point of B, and

$$[(\alpha * \delta) * \overline{(\beta * \delta)}] = [(\alpha * \delta) * (\bar{\delta} * \bar{\beta})] = [\alpha * \bar{\beta}),$$

which belongs to H by hypothesis.

Step 2. Topologizing E. One way to topologize E is to give *.P* the compact open topology and E the corresponding quotient topology. But we can topologize E directly as follows:

Let α be any element of *P* and let U be any path-connected neighborhood of $\alpha(1)$. Define

$B(U, \alpha) = \{(\alpha * \delta)^\# \mid \delta$ is a path in U beginning at $\alpha(1)\}$. Note that $\alpha^\#$ is an element of $B(U, \alpha)$, since if $b = \alpha(1)$, then $\alpha^\# = (\alpha * e_b)^\#$; this element belongs to $B(U, \alpha)$ by definition. We assert that the sets $B(U, \alpha)$ form a basis for a topology on E.

First, we show that if $\beta^{\#} \in B(U, \alpha)$ then $\alpha^{\#} \in B(U, \beta)$ and $B(U, \alpha) = B(U, \beta)$. If $\beta^{\#} \in B(U, \alpha)$, then $\beta^{\#} = (\alpha * \delta)^{\#}$ for some path δ in U. Then

$$(\beta * \bar{\delta})^{\#} = ((\alpha * \delta) * \bar{\delta})^{\#} \qquad \text{by (2)}$$
$$= \alpha^{\#} \qquad \text{by (1).}$$

so that $\alpha^{\#} \in B(U, \beta)$ by definition. See figure 3. We show first that $B(U, \beta) \subset B(U, \alpha)$. Note that the general element of $B(U, \beta)$ is of the form $(\beta * \gamma)^{\#}$, where γ is a path in U. Then note that

$$(\beta * \gamma)^{\#} = ((\alpha * \delta) * \gamma)^{\#}$$
$$= (\alpha * (\delta * \gamma))^{\#}$$

which belongs to $B(U, \alpha)$ by definition. Symmetry gives the inclusion $B(U, \alpha) \subset B(U, \beta)$ as well.

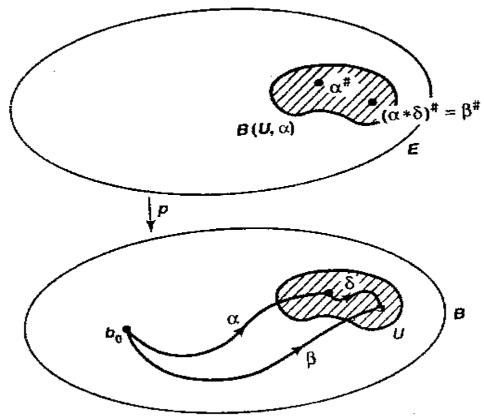

Figure 3

Now we show the sets $B(U, \alpha)$ form a basis. If $\beta^{\#}$ belongs to the intersection $B(U_1 \cdot \alpha_1) \cap B(U_2 \cdot \alpha_2)$ we need merely choose a path-connected neighborhood V of $\beta(1)$ contained in $U_1 \cap U_2$. The inclusion

$$B(V, \beta) \subset B(U_1 \cdot \beta) \cap B(U_2, \beta)$$

follows from the definition of these sets, and the right side of the equation equals $B(U_1, \alpha_1) \cap B(U_2, \alpha_2)$ by the result just proved.

Step 3. The map p is continuous and open. It is easy to see that *p* is open, for the image of the basis element $B(U, \alpha)$ is the open subset U of B : Given $x \in U$, we choose a path δ in U from $\alpha(1)$ to x; then $(\alpha * \delta)^{\#}$ is in $B(U, \alpha)$ and $p((\alpha * \delta)^{\#}) = x$.

To show that *p* is continuous, let us take an element $\alpha^{\#}$ of E and a neighborhood W of $p(\alpha^{\#})$. Choose a path-connected neighborhood U of the point $p(\alpha^{\#}) = \alpha(1)$ lying in W. Then $B(U, \alpha)$ is a neighborhood of $\alpha^{\#}$ that *p* maps into W. Thus *p* is continuous at $\alpha^{\#}$.

Step 4. Every point of B has a neighborhood that is evenly covered by *p*. Given $b_1 \in B$, choose U to be a path-connected neighborhood of b_1 that satisfies the further condition that the homomorphism $\pi_1(U, b_1) \rightarrow \pi_1(B, b_1)$ induced by inclusion is trivial. We assert that U is evenly covered by *p*.

First, we show that $p^{-1}(U)$ equals the union of sets $B(U, \alpha)$, as α ranges over all paths in B from b_0 to b_1. Since *p* maps each set $B(U, \alpha)$ onto U, it is clear that $p^{-1}(U)$ contains this union. On the other hand, if $B^{\#}$ belongs to $p^{-1}(U)$, then $\beta(1) \in U$. Choose a path δ in U from b_1 to $\beta(1)$ and let α be the path $\beta * \bar{\delta}$ from b_0 to b_1. Then $[\beta] = [\alpha * \delta]$, so that $\beta^{\#} = (\alpha * \delta)^{\#}$, which belongs to B(U, α). Thus $p^{-1}(U)$ is contained in the union of the sets B(U, α).

Second, note that distinct sets B(U, α) are disjoint. For if $B^{\#}$ belongs to B (U, α_1)$\cap$ B(U, α_2), then B(U, α_1) = B(U, β) = B(U, α_2) by step 2.

Third, we show *p* defines a bijective map of B(U, α) with U. It follows that p|B (U, α) is a homeomorphism, being bijective and continuous and open. We already know that *p* maps B(U, α) onto U. To prove injectivit, suppose that

$$p((\alpha * \delta)^{\#}) = p((\alpha * \delta)^{\#}),$$

where δ_1 and δ_2 are paths in U. Then δ_1 (1) = δ_2 (1). Because the homomorphism π_1 (U, b_1) $\rightarrow$ (B, b_1) induced by inclusion is trivial, $\delta_1 * \bar{\delta}_2$ is path homotopic in B to the constant loop. Then $[\alpha * \delta_1] = [\alpha * \delta_2]$, so that $(\alpha * \delta_1)^{\#}$ = (α = δ_2)#, as desired.

It follows that *p* : E $\rightarrow$ B is a covering map in the sense used in earlier chapters. To show it is a covering map in the sense used in this chapter, we must show E is path connected. This we shall do shortly.

Step 5. Lifting α path in B. Let e_0 denote the equivalence class of the constant path at b_0; then $p(e_0) = b_0$ by definition. Given a path α in B beginning at b_0, we calculate its lift to a path in E beginning at e_0 and show that this lift ends at $\alpha^{\#}$.

To begin, given c $\in$ [0, 1] let $\alpha_0 : I \rightarrow B$ denote the path defined by the equation

$$\alpha_0(t) = \alpha(tc) \quad \text{for} \quad 0 \leq t \leq 1.$$

Then α_0 is the "portion" of α that runs from $\alpha(0)$ to $\alpha(c)$. In particular, α_0 is the constant path at b_0, and α_1 is the path α itself. We define $\bar{\alpha} : I \rightarrow E$ by the equation

$$\bar{\alpha}(c) = (\alpha_0)^{\#}$$

and show that $\bar{\alpha}$ is continuous. Then $\bar{\alpha}$ is a lift of α, since $p(\bar{\alpha}(c)) = \alpha_c(1) = \alpha(c)$; furthermore, $\bar{\alpha}$ begins at $(\alpha_0)^{\#} = e_0$ and ends at $(\alpha_1)^{\#} = \alpha^{\#}$.

To verify continuity, we introduce the following notation. Given $0 \leq c < d \leq 1$, let $\delta_{c,d}$ denote the path that equals the positive linear map of I onto [c, d] followed by α. Note that the paths α_d and $\alpha_c *_{c,d}$ are path homotopic because one is just a reparmetrization of the other. See Figure 4.

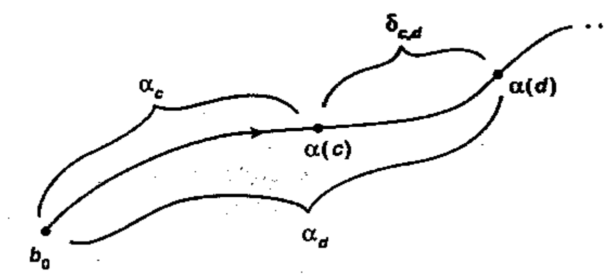

Figure 4

We now verify continuity of $\bar{\alpha}$ at the point c of [0, 1]. Let w be a basis element in E about the point $\bar{\alpha}(c)$. Then W equals $B(U, \alpha_c)$ for some path-connected neighborhood U of $\alpha(c)$. Choose $\in > 0$ so that for $|c - t| < \in$, the point $\alpha(t)$ lies in U. We show that if d is a point of [0, 1] with $|c - d| < \in$, then $\bar{\alpha}(d) \in W$; this proves continuity of $\bar{\alpha}$ at c. So suppose $|c - d| < \in$. Take first the case where d > c. Set $\delta_{c,\,d}$; then since $[\alpha_d] = [\alpha_c * \delta]$, we have

$$\bar{\alpha}(d) = (\alpha d)^{\#} = (\alpha_c * \delta)^{\#}.$$

Since δ lies in U, we have $\bar{\alpha}(d) \in B(U, \alpha_c)$ as desired. If d < c, set $\delta = \delta_{d,\,c}$ and proceed similarly.

Step 6. The map $p : E \rightarrow B$ is a covering map. We need only verify that E is path connected, and this is easy. For if $\alpha^{\#}$ is any point of E, then the lift $\bar{\alpha}$ of the path α is a path in E from e_0 to $\alpha^{\#}$.

Step 7. Finally, $H = p_*(\pi_1 E, e_0)$. Let α be a loop in B at b_0. Let $\bar{\alpha}$ be its lift to E beginning at e_0. Theorem 3.21 tells us that $[\alpha] \in p_* (\pi_1(E, e_0))$ if and only if $\bar{\alpha}$ is a loop in E. Now the final point of $\bar{\alpha}$ is the point $\alpha^{\#}$, and $\alpha^{\#} = e_0$ if and only if α is equivalent to the constant path at b_0, i.e., if and only if $[\alpha * \bar{e}_{b_0}] \in H$. This occurs precisely when $[\alpha] \in H$.

4. CONCLUSION

The paper deals with the important relationship between the topological spaces and the covering spaces which has been very well supported with many theorems and suitable examples.

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